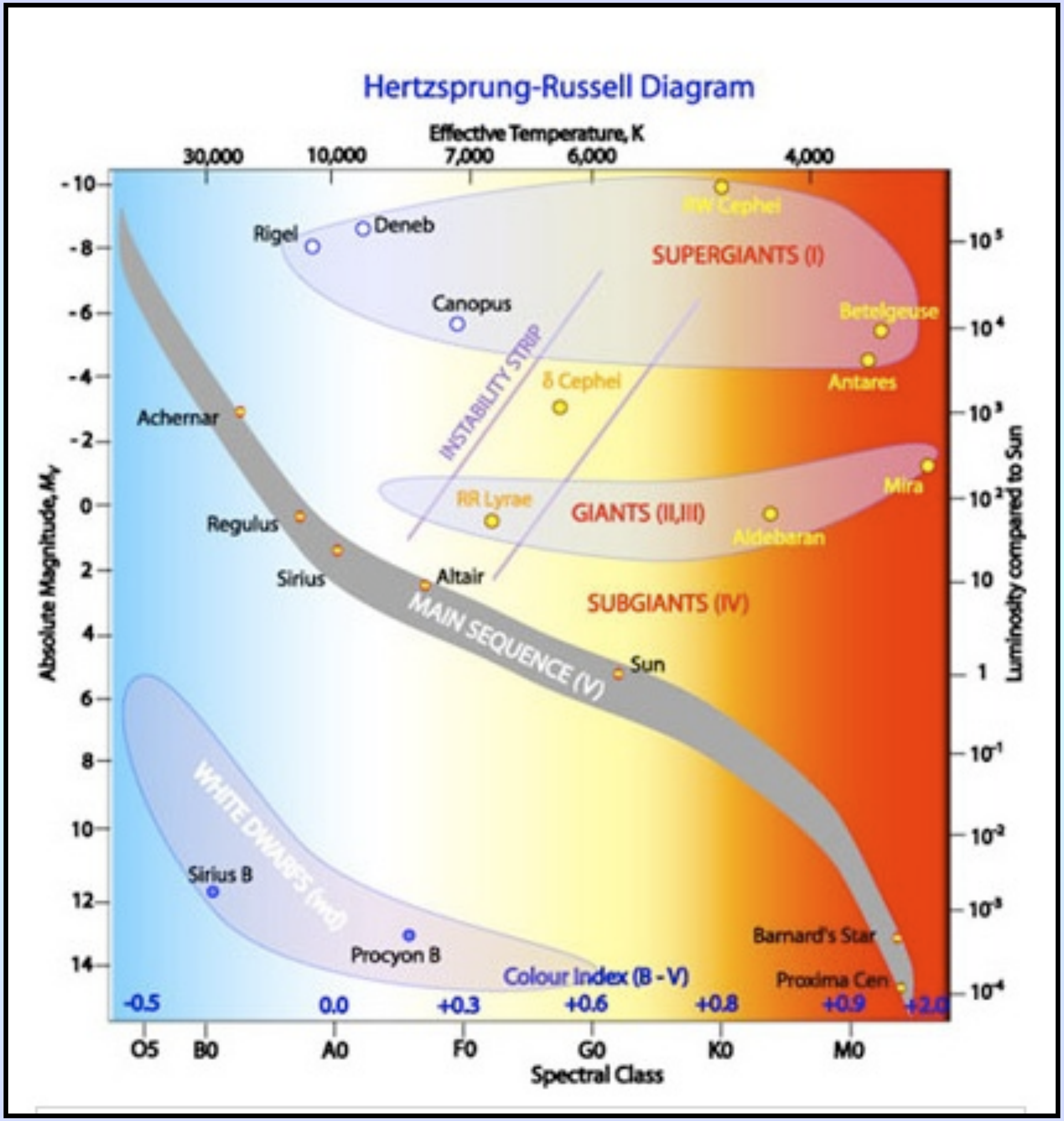
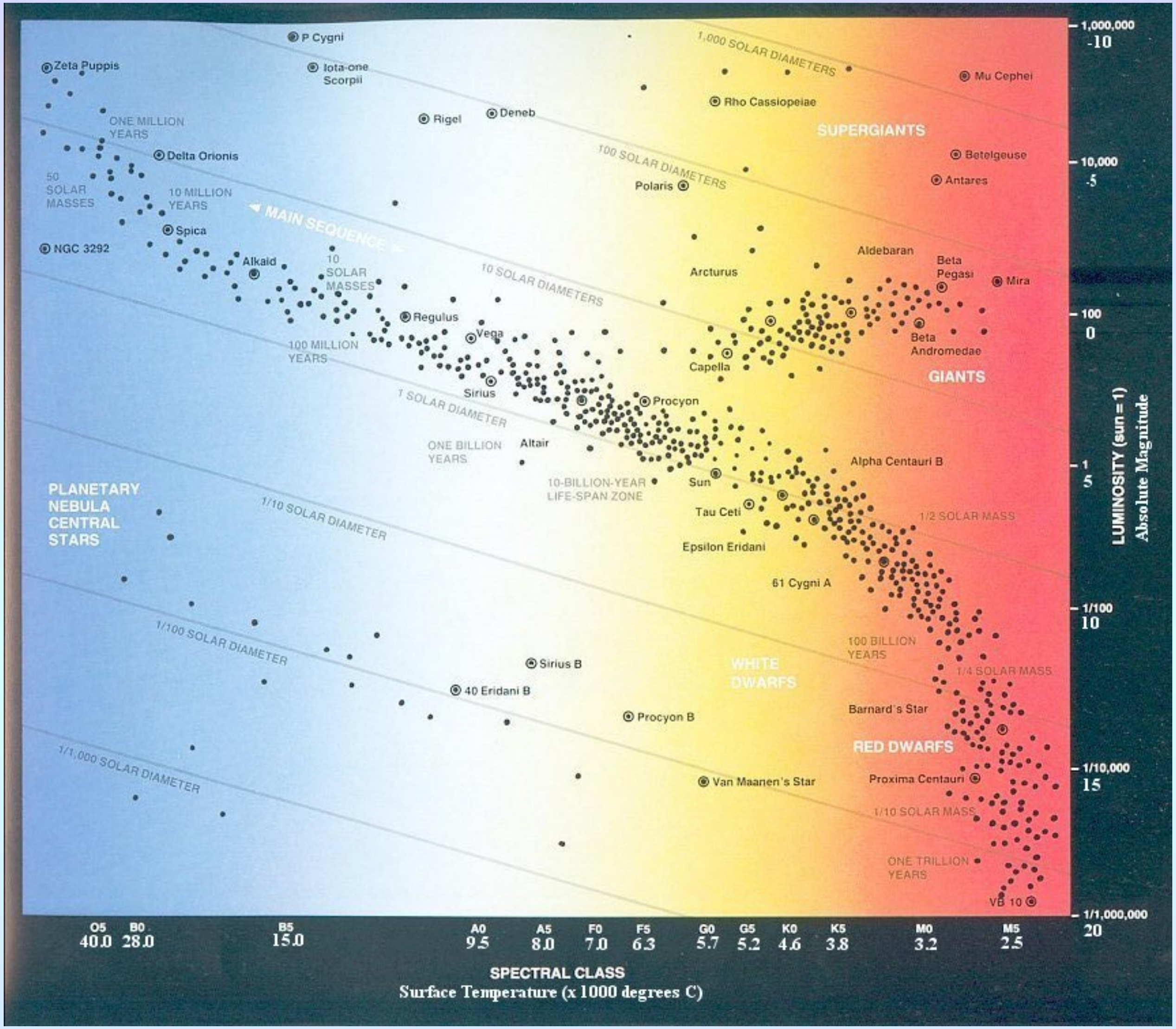
The HR Diagram

Most (>90%) stars lie on the "main sequence". A few stars are cool and extremely bright, so, by $L = 4 \pi R^2 \sigma T^4$, they must be extremely large. A few stars are hot, but extremely faint, so they must be very small.



The HR Diagram

Most (>90%) stars lie on the "main sequence". (But most stars you know are giant stars.) You can see them much further away!



$$f \propto \frac{L}{d^2} \quad \Rightarrow \quad L \propto d^2$$

$$N_{\mathrm{obj}} \propto V \propto d^3 \quad \Rightarrow \quad N_{\mathrm{obj}} \propto L^{3/2}$$

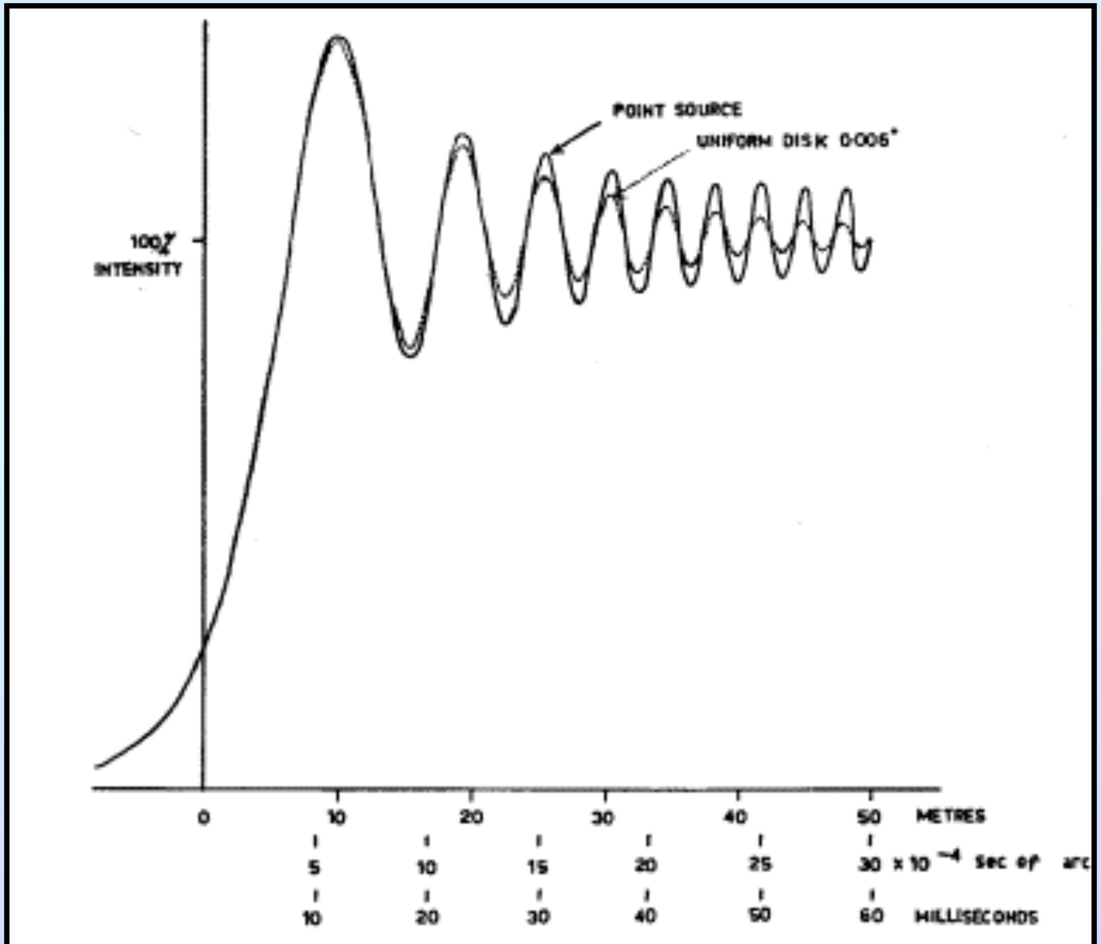
Determining Stellar Radii

At stellar distances, virtually all stars are unresolved, even with the *Hubble Space Telescope*. There are only a few methods that can (theoretically) produce measurements of stellar radii:

- Lunar occultations (for angular diameters)
  - Objects must be bright and within 5° of the ecliptic plane
- Interferometry (angular diameters)
  - Objects must be bright
  - Best in the IR, where the atmosphere is better behaved
- Baade-Wesselink (for pulsating stars)
- Eclipsing binary stars

Lunar Occultations

The Moon orbits the Earth in ~ 29.5 days, i.e., at a rate of ~ 0.5″/sec. At ~ 10 pc, the angular diameter of the Sun is ~ 0.001 arcsec, which translates into an occultation timescale of a few millisec.



Stars must be bright and within 5° of the ecliptic plane.

Table 7.3 *Some angular diameters measured from lunar occultations according to Ridgeway, Wells and Joyce (1977)*

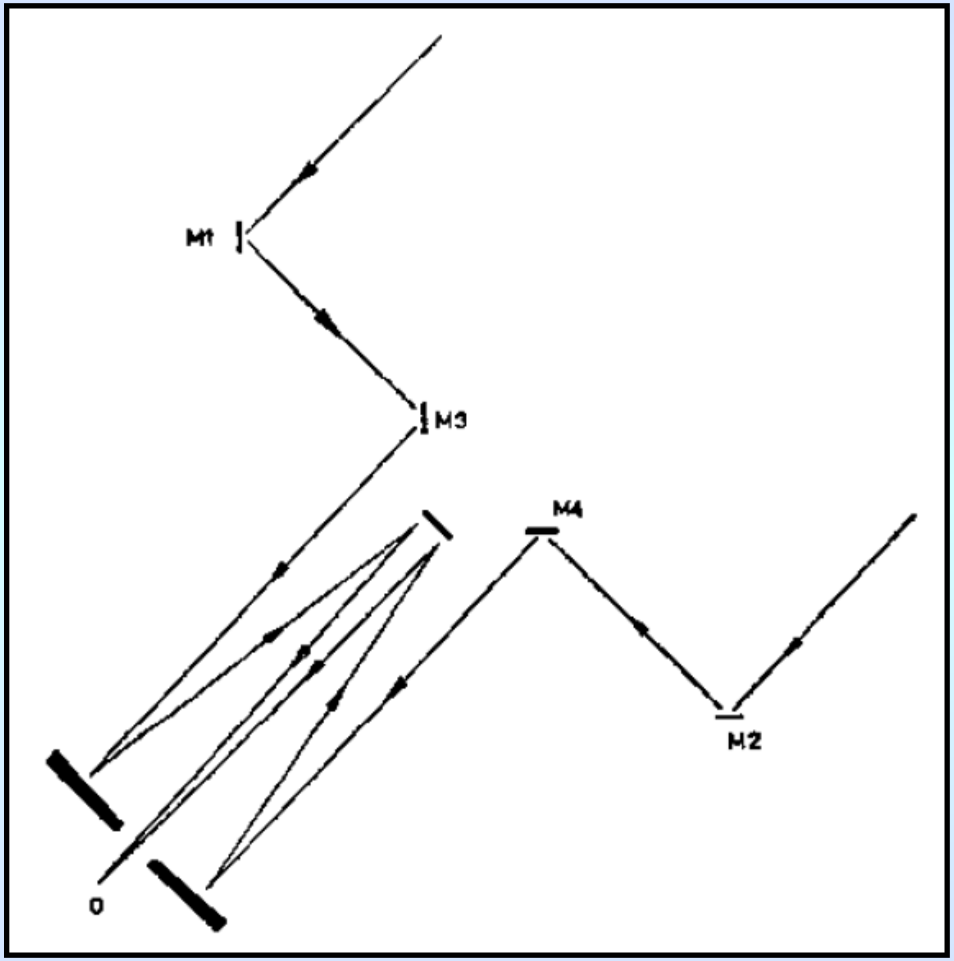
HR No.[2]	Star	Sp. type[3]	B – V	m_V	Θ_{LD}[1] 10^{-3} arcsec	π arcsec	$R/R_\odot$	T_{eff}
2286	μ Gem	M3III	1.64	2.88	13.65	0.020	73.2	3650
867	RZ Ari	M6III	1.47	5.91	10.18	0.014	78.0	3160
1977	Y Tau	C5II	3.03	6.95	8.58	?	?	?
5301		M2III	1.72	4.91	4.41	0.012	39.4	4040[4]
4902	ψ Vir	M3III	1.60	4.79	5.86	0.021	29.9	3530
7150	ξ^2 Sgr	K1III	1.18	3.51	3.80	0.011	37.1	4210
9004	TX Psc	C5II	2.60	5.04	9.31	–0.004	?	?
3980	31 Leo	K4III	1.45	4.37	3.55	?	?	4000
7900	υ Cap	M2III	1.66	5.10	4.72	0.019	26.6	3440

1. Considering the limb darkening of the disk.
2. Number in the Bright Star catalogue, Hoffleit and Jaschek (1982).
3. From the Bright Star catalogue
4. Assuming BC = 2.0.

Amplitude or Intensity Interferometry

Combines the light from two slits (or telescopes) whose separation can be adjusted; object size can be derived by observing the fringes as a function of image separation, or comparing the “noise” as the separation of the detectors increases.

Once again, the objects must be bright and large (supergiants)



Amplitude or Intensity Interferometry

Combines the light from two slits (or telescopes) whose separation can be adjusted; object size can be derived by observing the fringes as a function of image separation, or comparing the "noise" as the separation of the detectors increases.

Once again, the objects must be bright and large (supergiants)

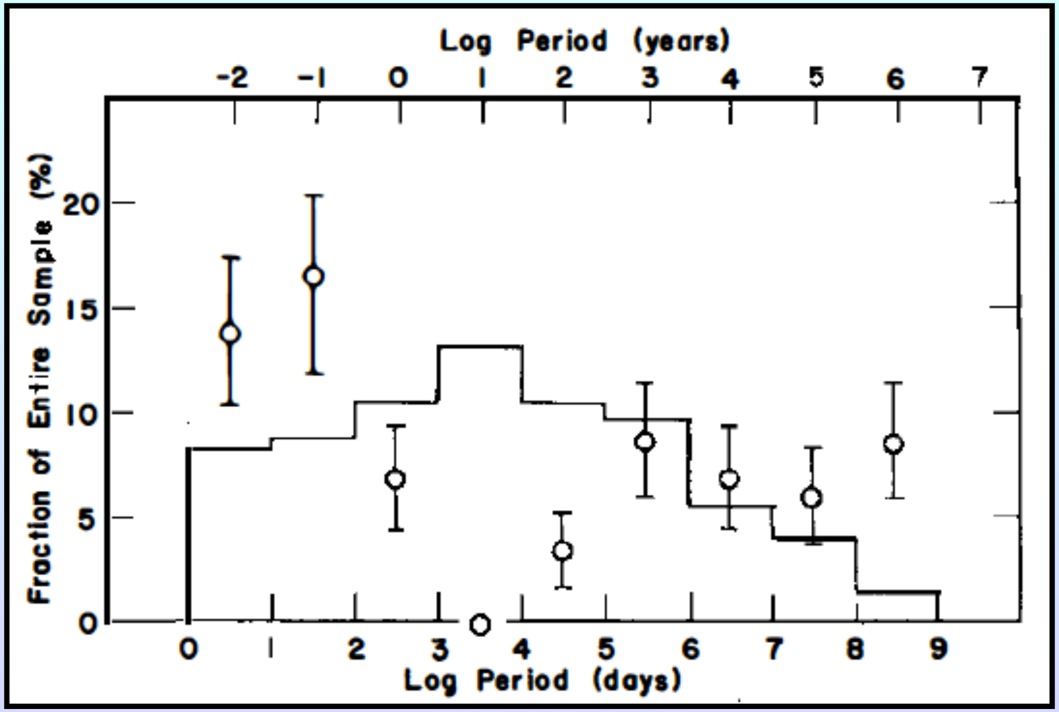
Table 7.2 *Angular diameters, θ, measured with the Hanbury Brown interferometer*

Star[1] number	Star name	Type[2]	Angular diameter[3] in 10^{-3}s of arc $\Theta_{LD} \pm \sigma$	Temperature[4] $[T_{eff} \pm e]$/K
472	α Eri	B3(Vp)	1.92 ± 0.07	$13\,700 \pm 600$
1713	β Ori	B8(Ia)	2.55 ± 0.05	$11\,500 \pm 700$
1790	γ Ori	B2 (III)	0.72 ± 0.04	$20\,800 \pm 1300$
1903	ϵ Ori	B0(Ia)	0.69 ± 0.04	$24\,500 \pm 2000$
1948	ζ Ori	O9.5(Ib)	0.48 ± 0.04	$26\,100 \pm 2200$
2004	κ Ori	B0.5 (Ia)	0.45 ± 0.03	$30\,400 \pm 2000$
2294	β CMa	B1(II-III)	0.52 ± 0.03	$25\,300 \pm 1500$
2326	α Car	F0 (Ib-II)	6.6 ± 0.8	7500 ± 250
2421	γ Gem	A0(IV)	1.39 ± 0.09	9600 ± 500
2491	α CMa	A1(V)	5.89 ± 0.16	$10\,250 \pm 150$
2618	ϵ CMa	B2 (II)	0.80 ± 0.05	$20\,800 \pm 1300$
2693	δ CMa	F8 (Ia)	3.60 ± 0.50	…
2827	η CMa	B5 (Ia)	0.75 ± 0.06	$14\,200 \pm 1300$
2943	α CMi	F5 (IV-V)	5.50 ± 0.17	6500 ± 200
3165	ζ Pup	O5(f)	0.42 ± 0.03	$30\,700 \pm 2500$
3207	γ^2 Vel	WC8 + O9	0.44 ± 0.05	$29\,000 \pm 3000$
3685	β Car	A1(IV)	1.59 ± 0.07	9500 ± 350
3982	α Leo	B7(V)	1.37 ± 0.06	$12\,700 \pm 800$
4534	β Leo	A3(V)	1.33 ± 0.10	9050 ± 450
4662	γ Crv	B8(III)	0.75 ± 0.06	$13\,100 \pm 1200$
4853	β Cru	B0.5(III)	0.722 ± 0.023	$27\,900 \pm 1200$
5056	α Vir	B1(IV)	0.87 ± 0.04	$22\,400 \pm 1000$
5132	ϵ Cen	B1(III)	0.48 ± 0.03	$26\,000 \pm 1800$
5953	δ Sco	B0.5(IV)	0.46 ± 0.04	…
6175	ζ Oph	O9.5(V)	0.51 ± 0.05	…
6556	α Oph	A5(III)	1.63 ± 0.13	8150 ± 400
6879	ϵ Sgr	A0(V)	1.44 ± 0.06	9650 ± 400
7001	α Lyr	A0(V)	3.24 ± 0.07	9250 ± 350
7557	α Aql	A7 (IV, V)	2.98 ± 0.14	8250 ± 250
7790	α Pav	B2.5(V)	0.80 ± 0.05	$17\,100 \pm 1400$
8425	α Gru	B7(IV)	1.02 ± 0.07	$14\,800 \pm 1200$
8728	α PsA	A3(V)	2.10 ± 0.14	9200 ± 500

1. Bright star catalog number (Hoffleit and Jaschek, 1982).
2. Spectral type and luminosity class (in brackets), to be discussed in Chapter 10.
3. True angular diameter allowing for the effects of limb-darkening.
4. Effective temperatures will be discussed in Chapter 8.

Binary Stars

- The most direct measurements of stellar radii (and stellar masses) come from the analysis of binary stars.
- Perhaps ~ 85% of all stars in the Milky Way are part of multiple systems (binaries, triplets or more)
- The periods (separations) of these binaries span the entire range of possibilities, from contact binaries to separations of ~ 0.1 pc. The distribution of separations appears roughly flat in log-space.



Binary Stars

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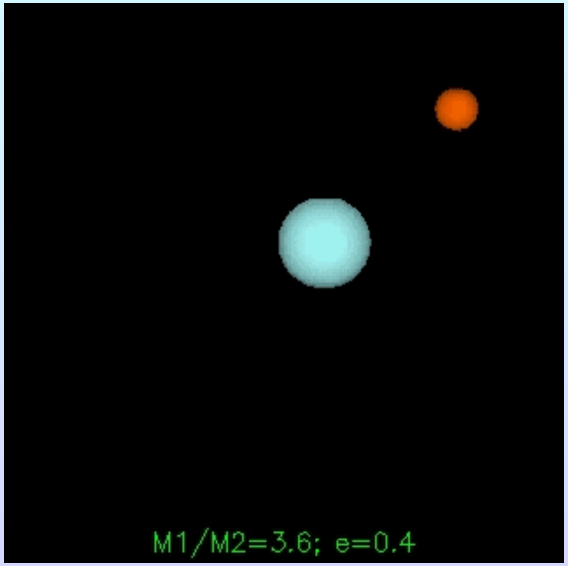
- Perhaps ~ 85% of all stars in the Milky Way are part of multiple systems (binaries, triplets or more)
- The periods (separations) of these binaries span the entire range of possibilities, from contact binaries to separations of ~ 0.1 pc. The distribution of separations appears roughly flat in log-space.
- The motions of binary stars are controlled by Kepler's laws
    - Orbits are ellipses with a star at a focus
    - Orbits sweep out equal areas in equal times
    - $(M_1 + M_2)\, P^2 = a^3$
- The separations and motions of the two stars in the center-of-mass frame is given by

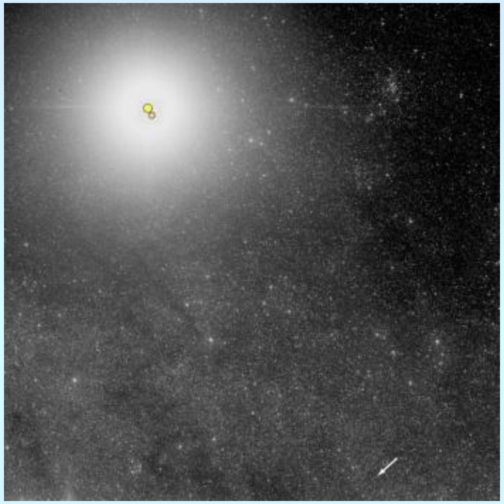


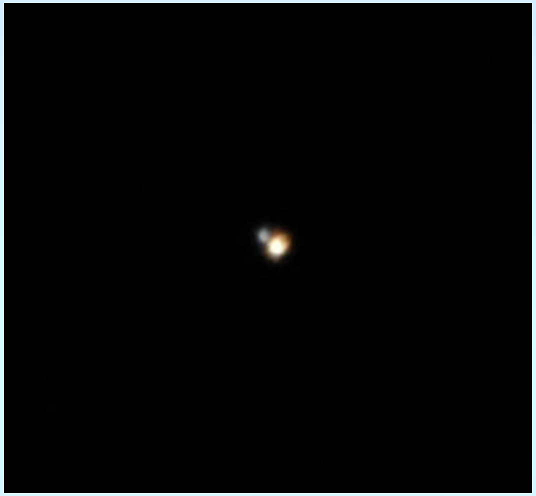
$$\boxed{\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1} = \frac{v_2}{v_1}}$$

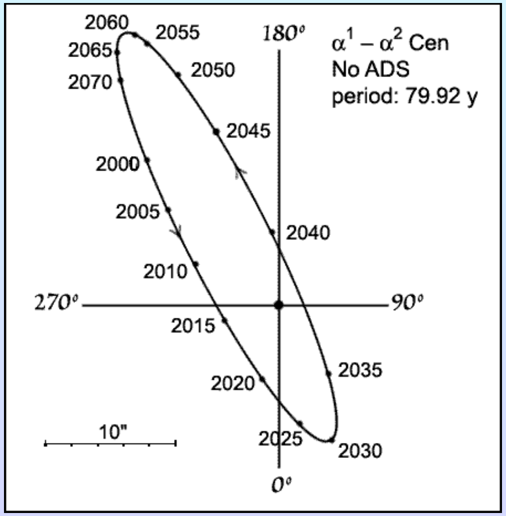
Binary Stars

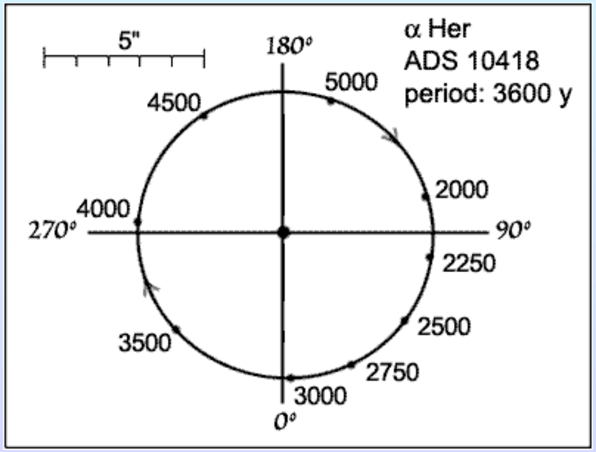
There are several types of binaries:

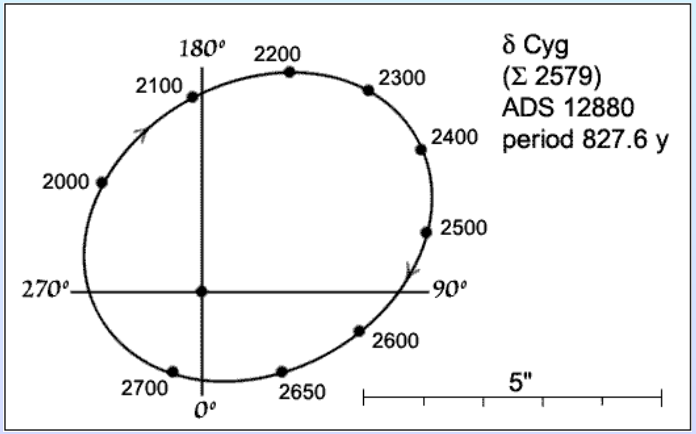
- Visual







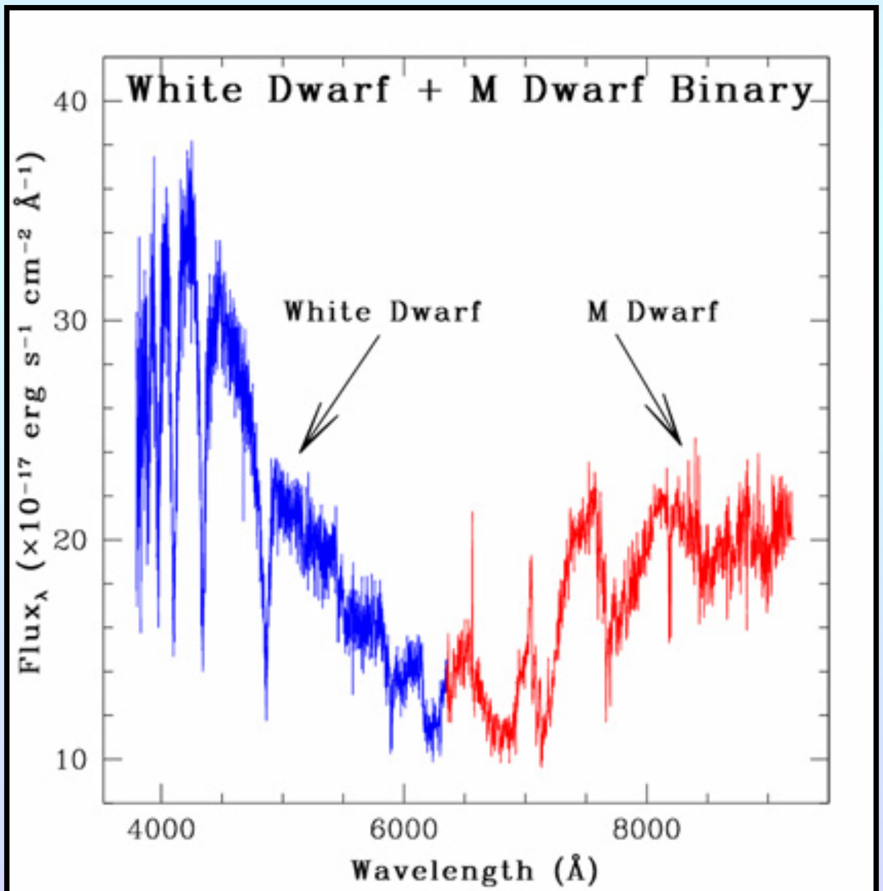


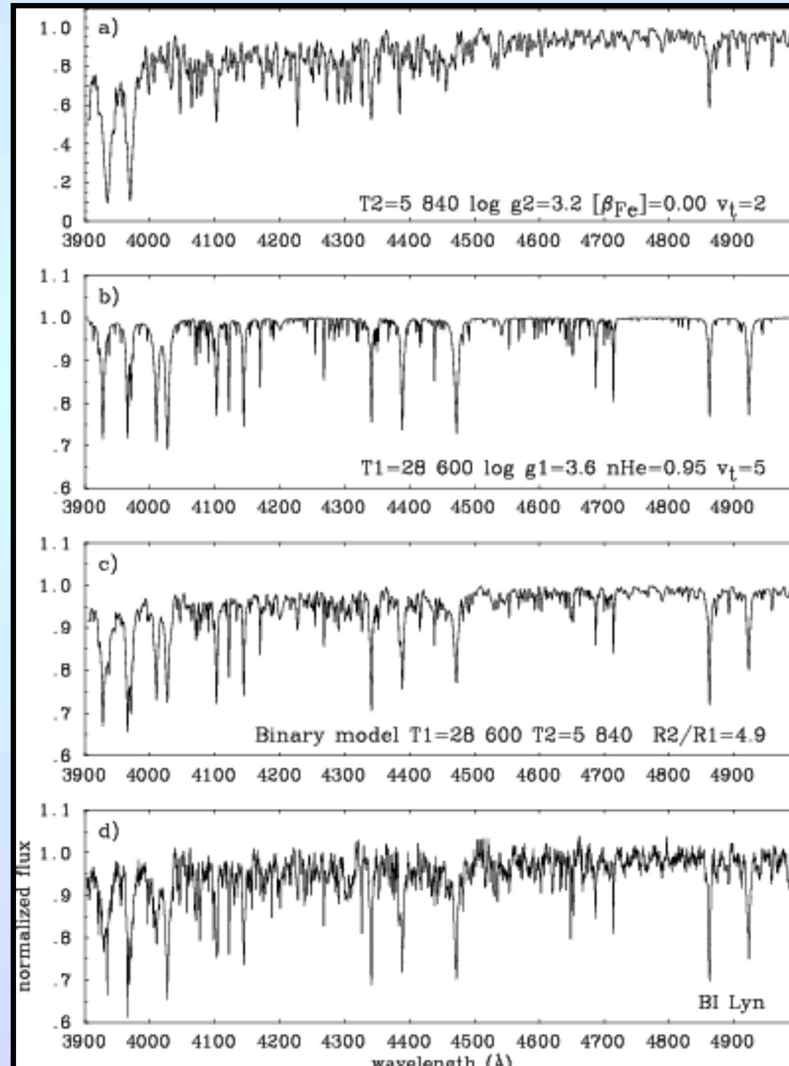


Binary Stars

There are several types of binaries:

- Visual
- Spectrum

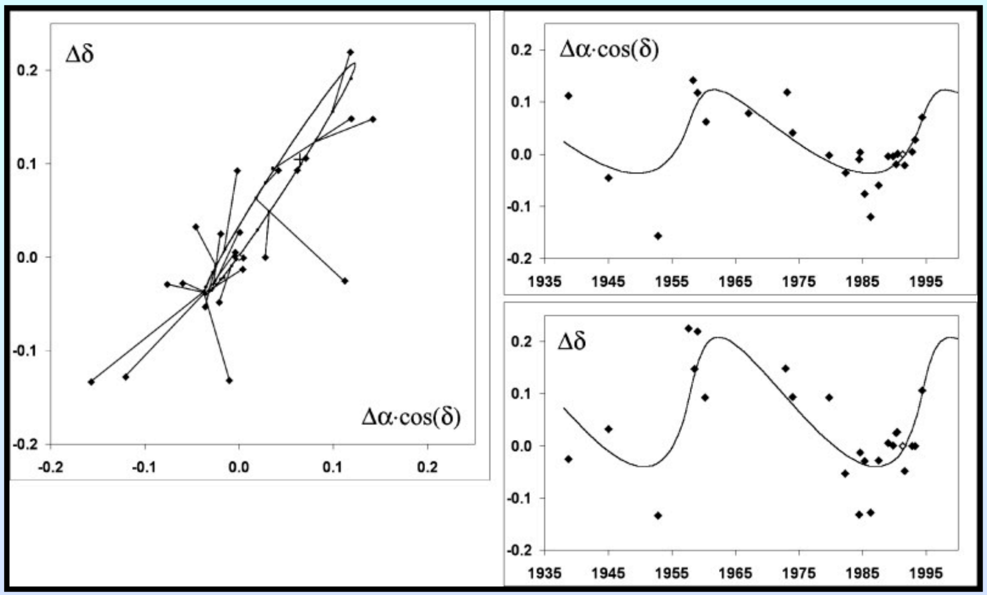


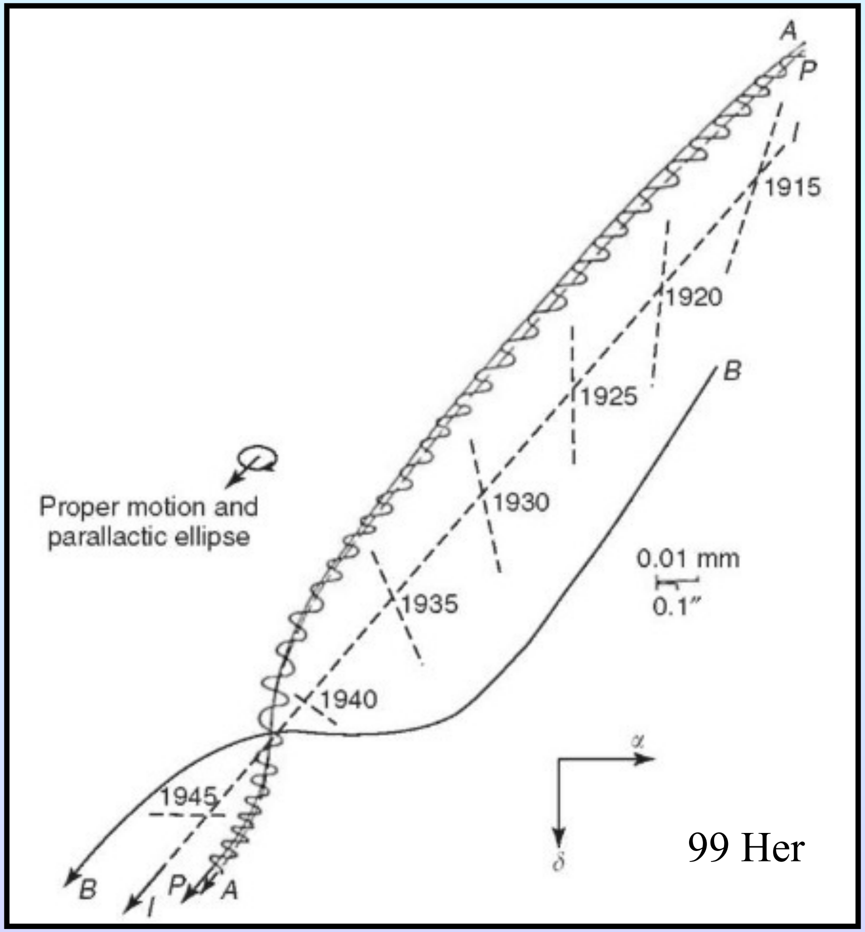


Binary Stars

There are several types of binaries:

- Visual
- Spectrum
- Astrometric

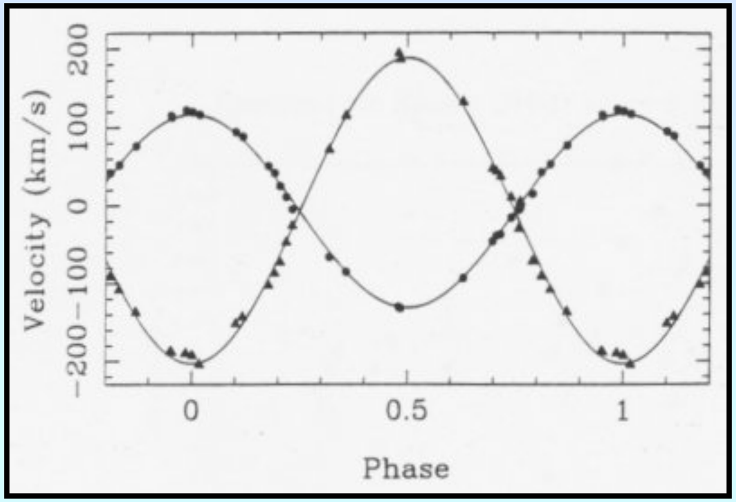


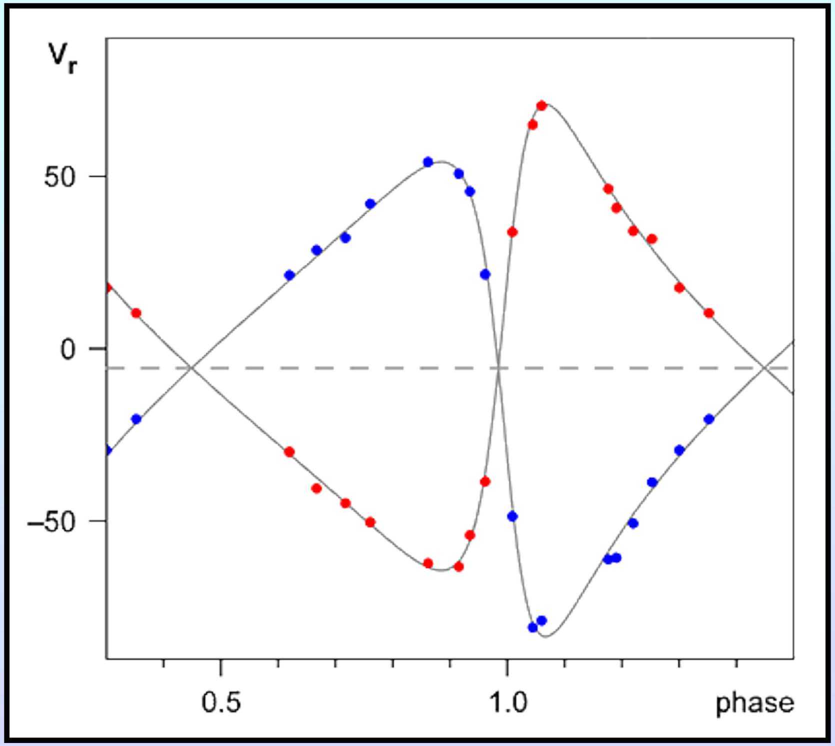


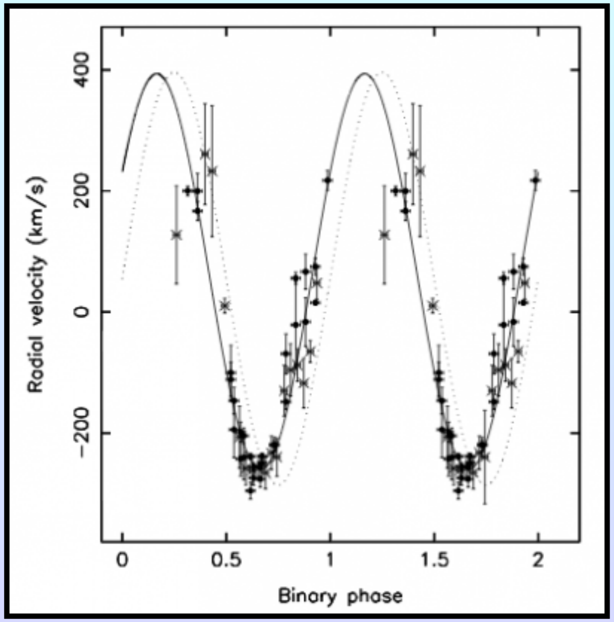
Binary Stars

There are several types of binaries:

- Visual
- Spectrum
- Astrometric
- Spectroscopic



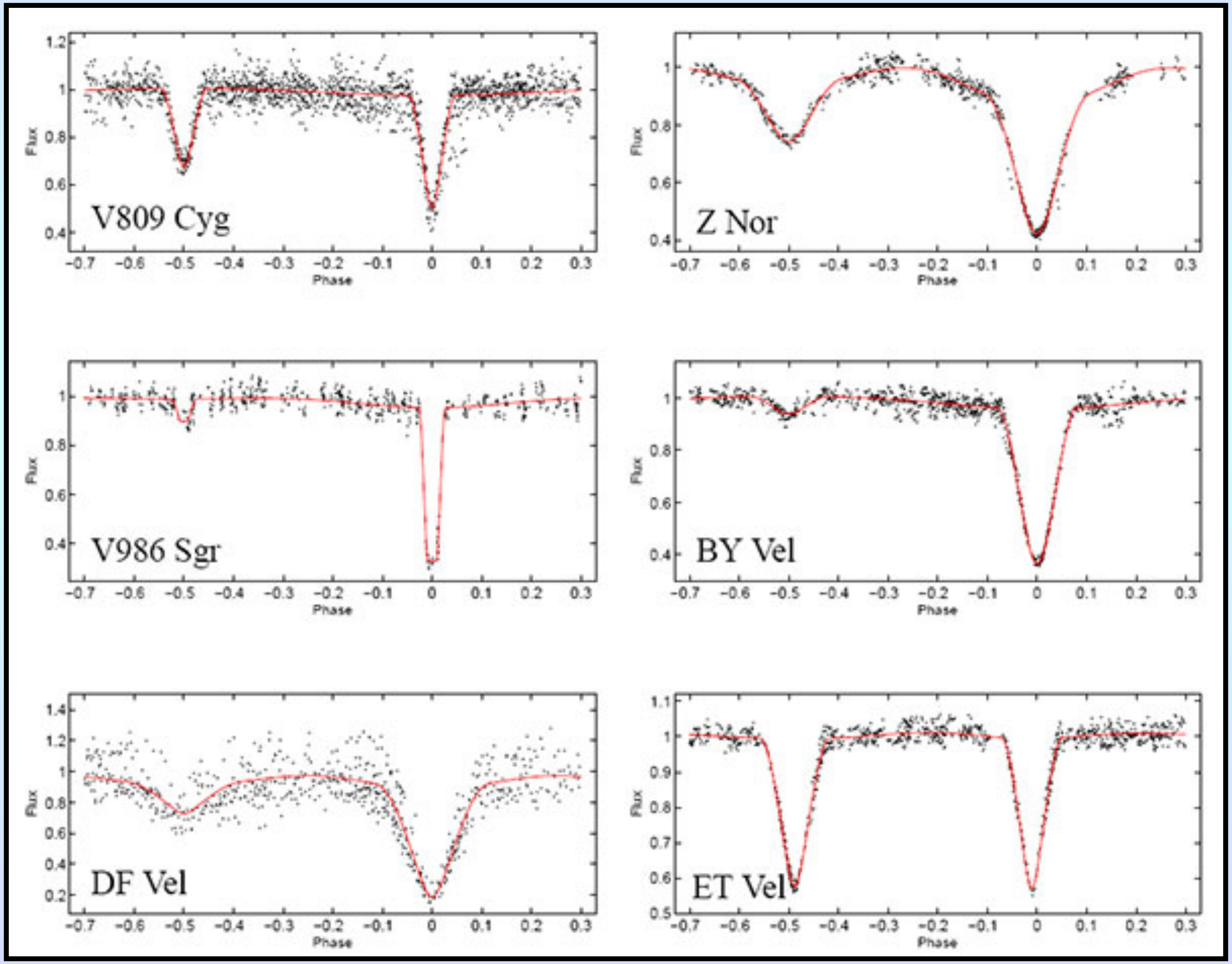




Binary Stars

There are several types of binaries:

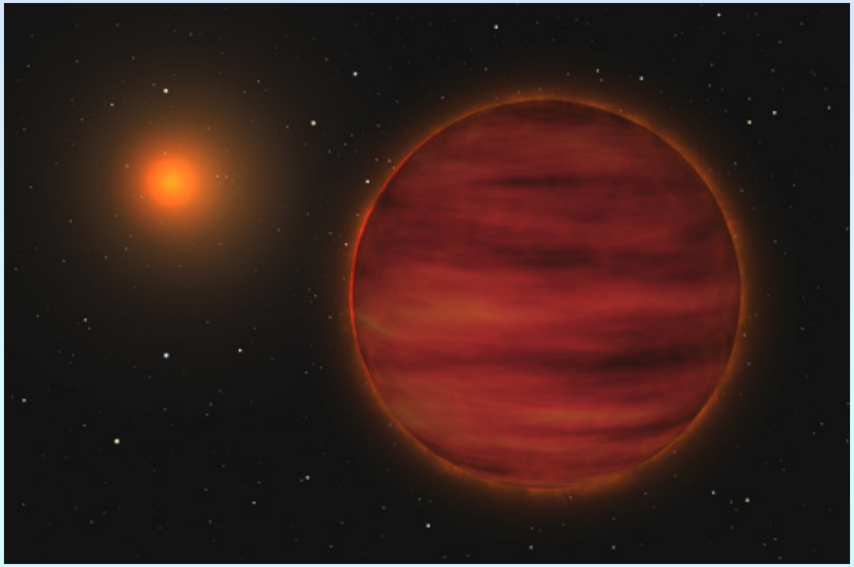
- Visual
- Spectrum
- Astrometric
- Spectroscopic
- Eclipsing

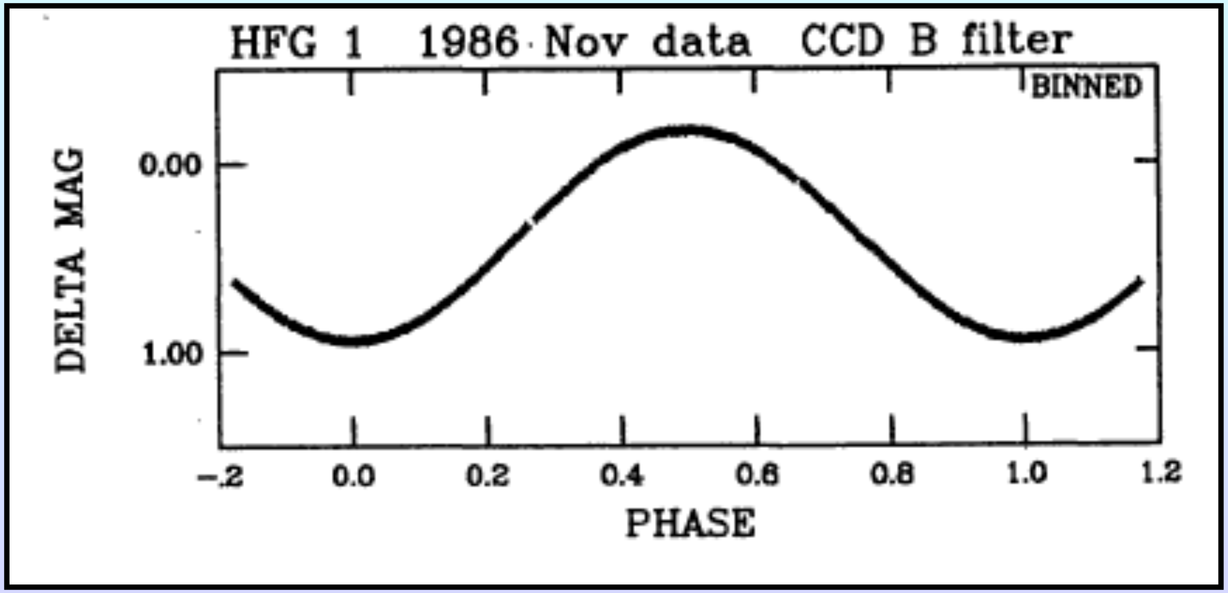


Binary Stars

There are several types of binaries:

- Visual
- Spectrum
- Astrometric
- Spectroscopic
- Eclipsing
- Reflection

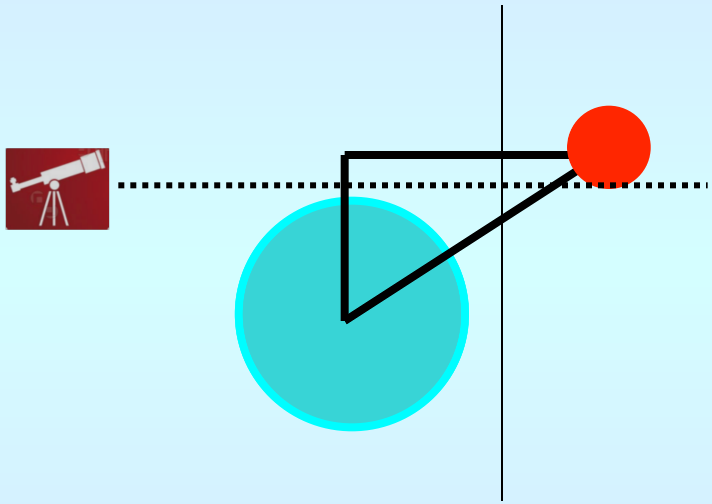


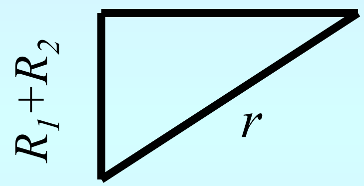


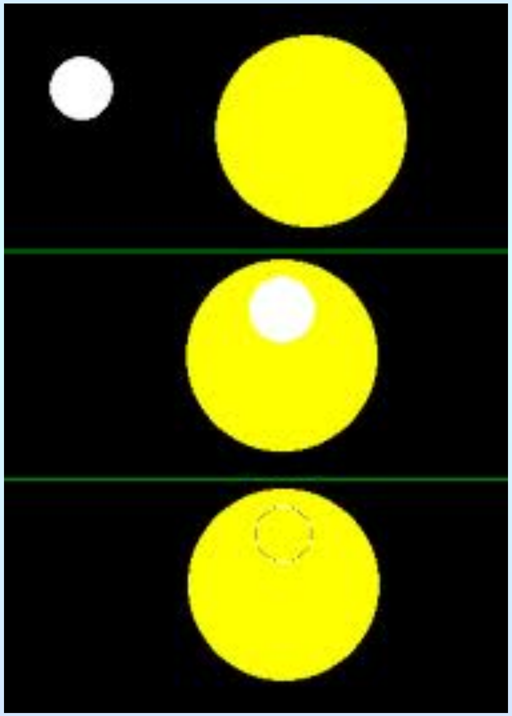
Eclipsing Binaries

If a binary system is eclipsing, then we have a good estimate of its inclination. If r is the separation of the two stars, then, for at least a partial eclipse

$$|\cos i| < \frac{R_1 + R_2}{r}$$







For a total eclipse to happen $$|\cos i| < \frac{R_1 - R_2}{r}$$

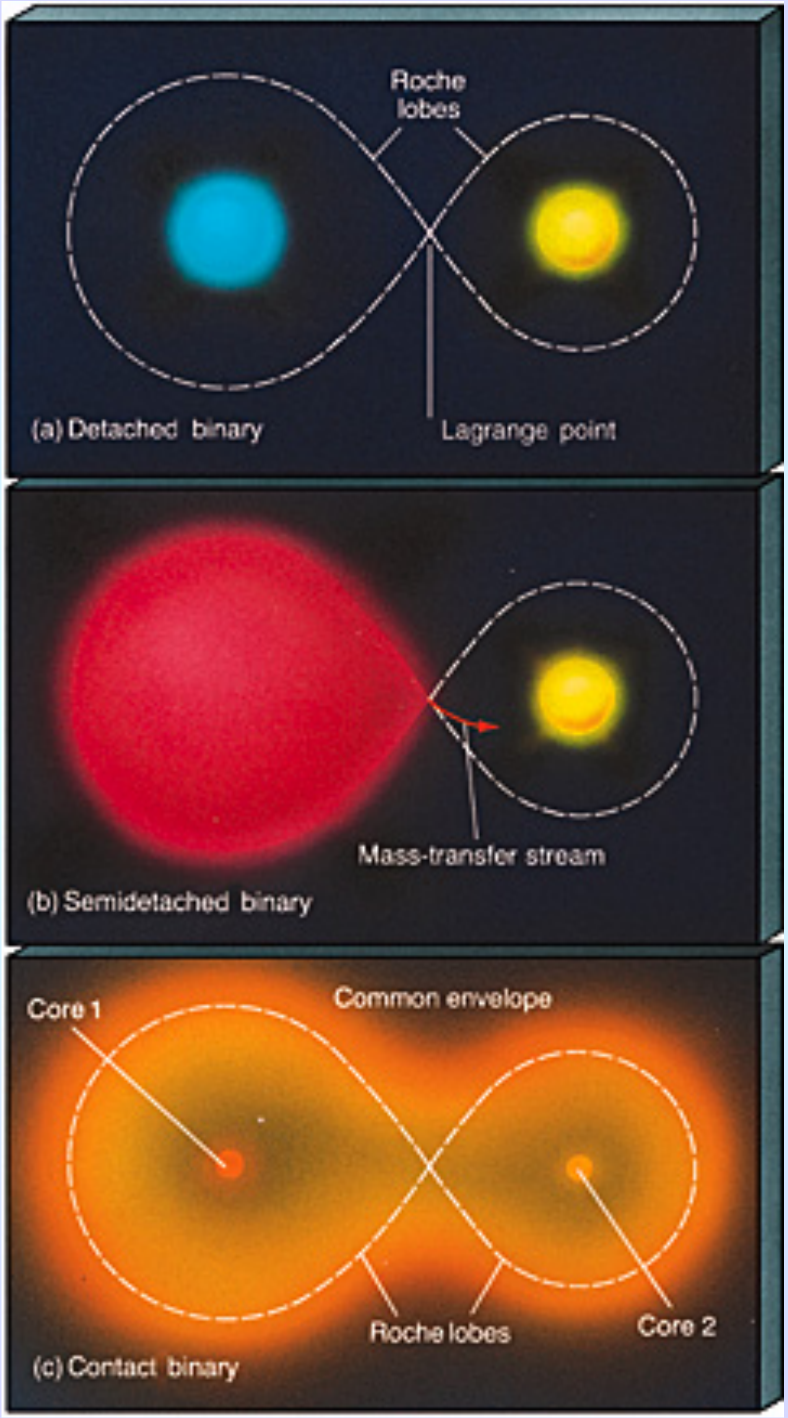
If r » R_1+R_2 then $i \sim 90°$. Also, the larger the separation, the rarer the phenomenon. Many eclipsing binaries have short periods.

If the separation becomes too small, the stars can become tidally distorted and/or interact

Detached: the stars are separate and do not affect one another.

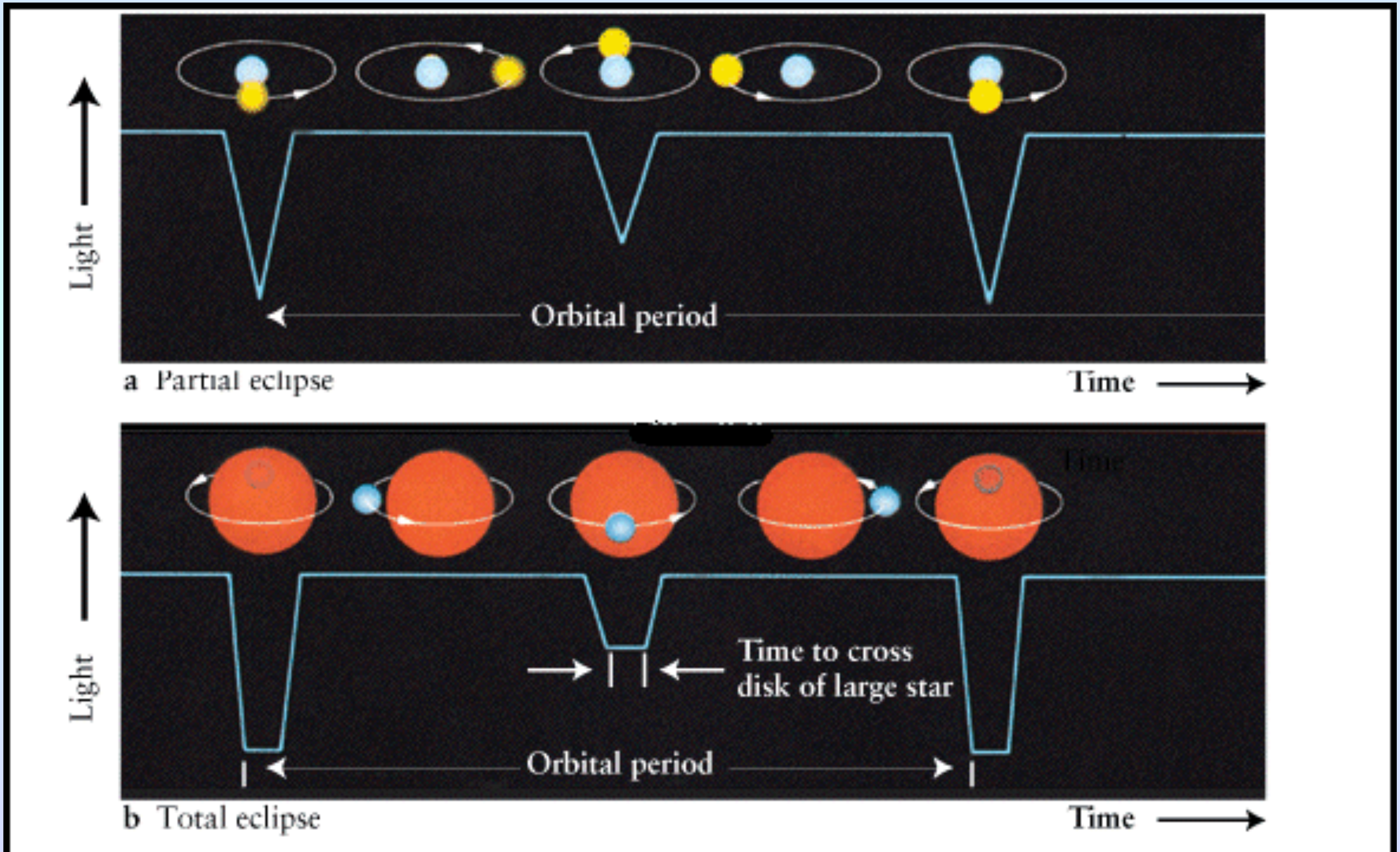
Semi-detached: one star is spilling mass (*i.e.,* accreting) onto the other

Contact: two stars are present inside a common envelope (*i.e.,* it is a common-envelope binary).



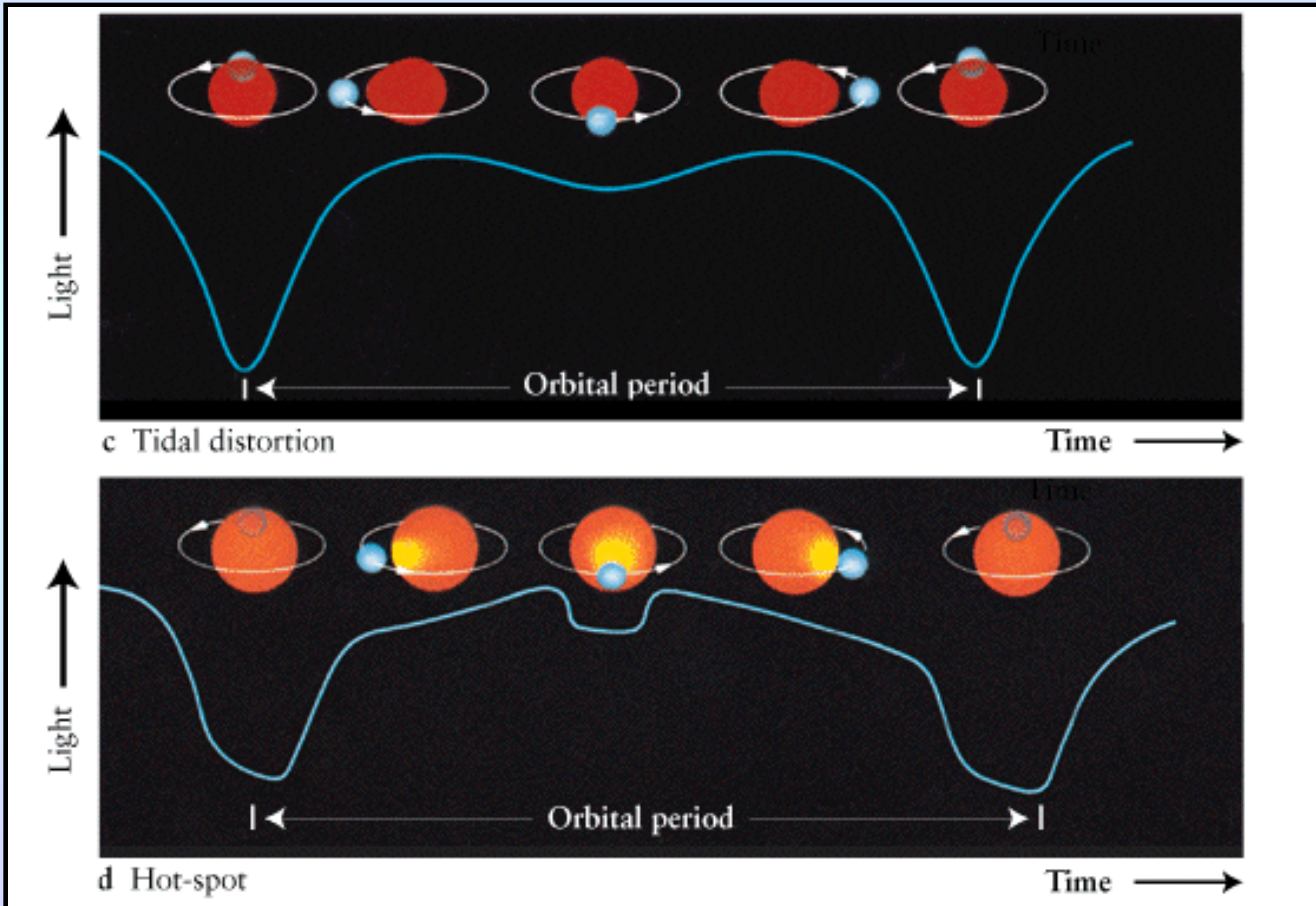
Partial and Total Eclipses

The shape of the light curve during eclipse defines whether the eclipse is total or partial.



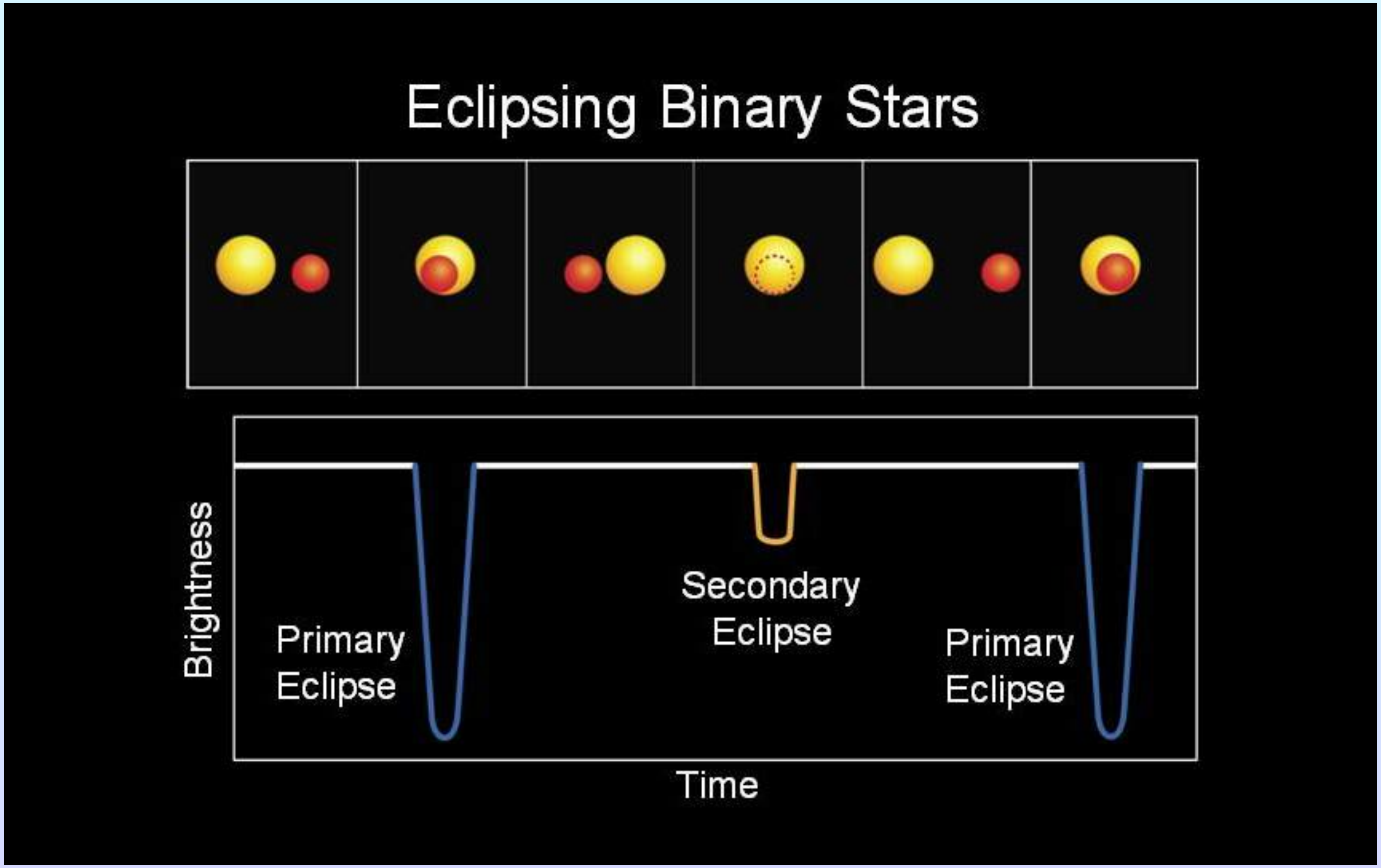
Partial and Total Eclipses

In practice, other effects, such as star-spots, tidal distortions, limb-darkening, and hot spots may also effect the shape of the light curve.



Depth of Eclipse

From symmetry the area blocked during the primary eclipse is exactly the same as that of the second eclipse. Consequently, when calculating the amount of light that is eclipsed, $L = 4\pi R^2 \sigma T^4$, the radius doesn't matter – only the temperature counts.



Total Eclipses

Let $F = 1$ be the relative flux from a system out of eclipse, $F_{T,}$ the relative flux during total eclipse, and F_{a} the flux during the annular eclipse. If we designate Star 1 to be the larger star, and Star 2 the smaller star, then

During the total eclipse

- $F_1 = F_T \quad \text{and} \quad F_2 = F - F_T$

During the annular eclipse

- $F_a = F_1 + F_2 - \left(\frac{R_2}{R_1}\right)^2 F_1 = 1 - \left(\frac{R_2}{R_1}\right)^2 F_1$

Therefore, the ratio of the stellar radii is simply

- $\kappa = \frac{R_2}{R_1} = \left(\frac{1 - F_a}{F_T}\right)^{1/2}$

Total Eclipses

For systems, with $i = 90°$, the eclipse begins when

$$\sin\theta_e = \frac{R_1 + R_2}{a}$$

and becomes full when

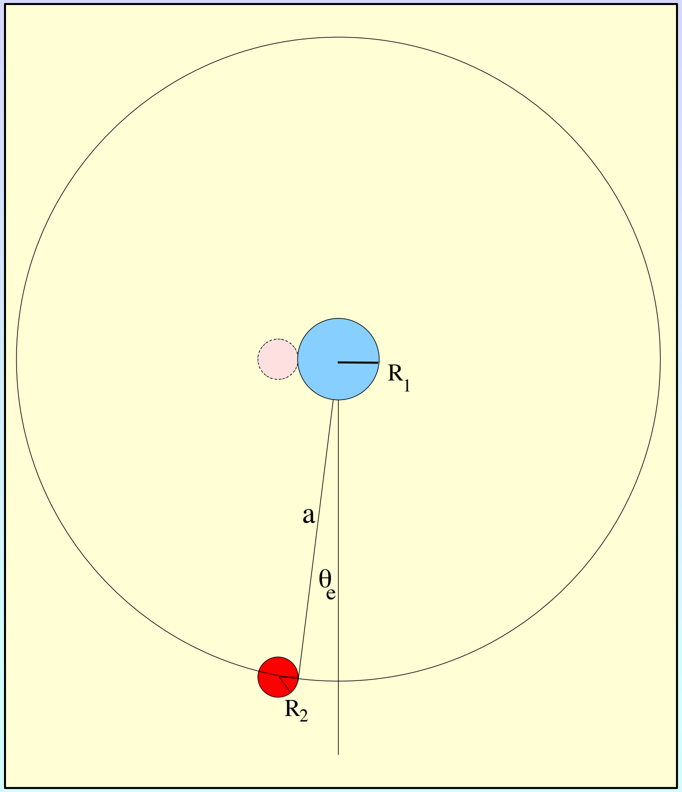
$$\sin\theta_i = \frac{R_1 - R_2}{a}$$

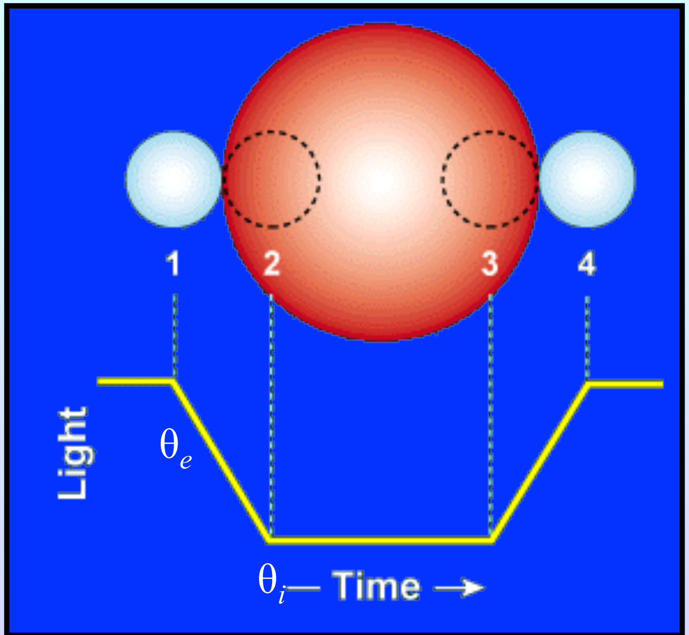
In the more general case ($i \neq 90°$), the eclipse beings when the projected separation is

$$\delta_e^2 = \sin^2\theta_e + \cos^2\theta_e \cos^2 i = \left(\frac{R_1 + R_2}{a}\right)$$

and is full when

$$\delta_i^2 = \sin^2\theta_i + \cos^2\theta_i \cos^2 i = \left(\frac{R_1 - R_2}{a}\right)$$





Solving Eclipsing Binaries

Note that for total eclipses, the relative sizes of the stars and the orbit, along with the inclination of the system, can be computed without any information other than the light curve:

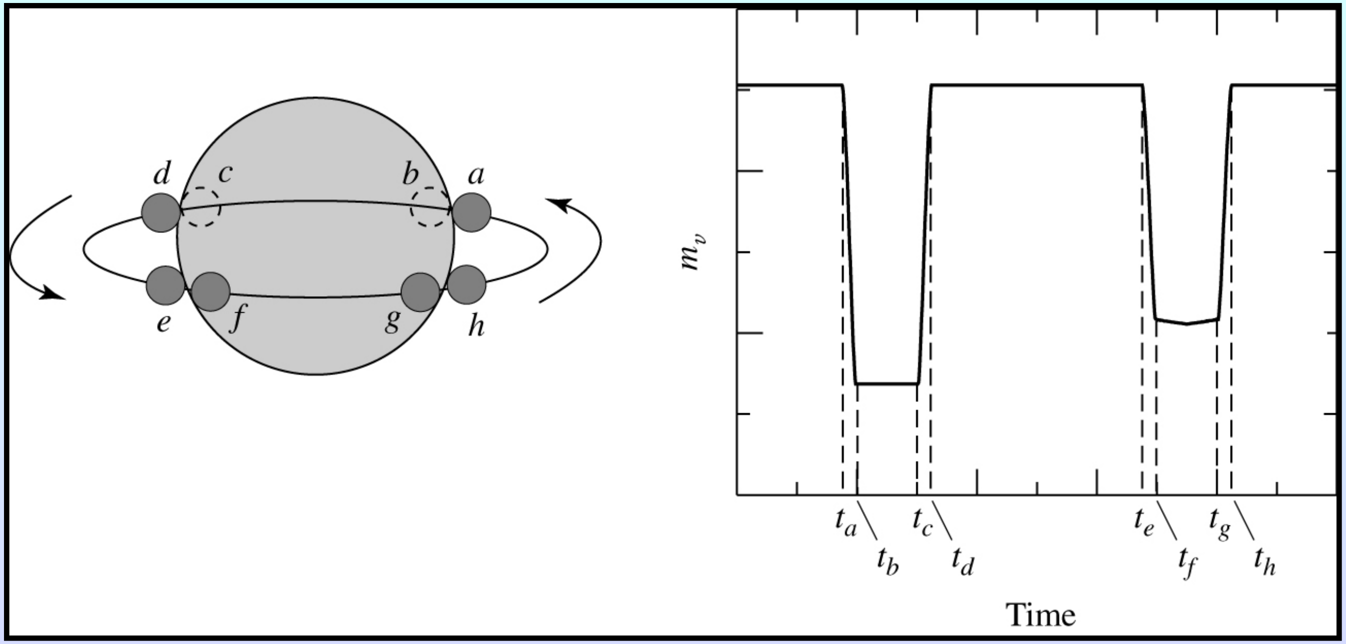
- Depth of total eclipse: $F_1 = 1 - F_2$
- The ratio of the stellar sizes: $\dfrac{R_2}{R_1} = \left(\dfrac{1 - F_a}{F_T}\right)^{1/2}$
- The angle of egress: $\sin^2\theta_e + \cos^2\theta_e \cos^2 i = \left(\dfrac{R_1 + R_2}{a}\right)$
- The angle of ingress: $\sin^2\theta_i + \cos^2\theta_i \cos^2 i = \left(\dfrac{R_1 - R_2}{a}\right)$

If you work with the quantities, R_2/R_1, R_2/a, i, and F_1/F_2, there are 4 equations and 4 unknowns.

Eclipsing Spectroscopic Binaries

If a system is both an eclipsing and spectroscopic binary, then everything can be measured for the system:

- From the eclipse, you can measure R_2/R_1, R_2/a, i, and F_1/F_2
- From the velocities, you can obtain absolute size. For example, if we assume $i \sim 90°$ and a circular orbit, then measuring individual stellar radii is simple.



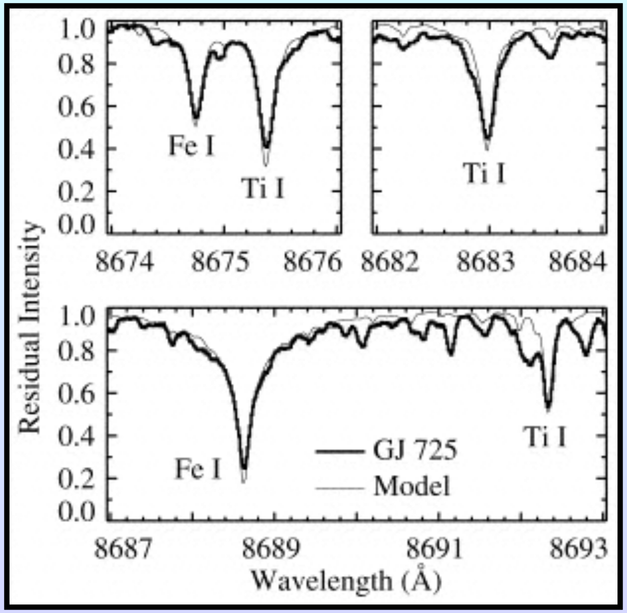
$$R_1 = \frac{v}{2}\left(t_c - t_a\right)$$

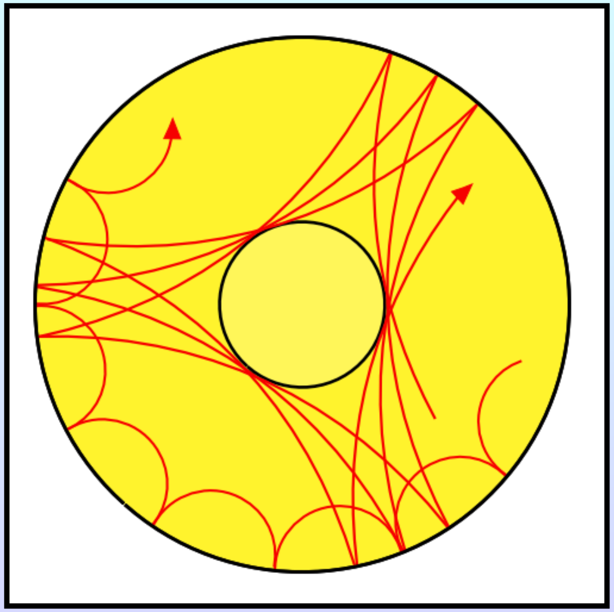
$$R_2 = \frac{v}{2}\left(t_b - t_a\right)$$

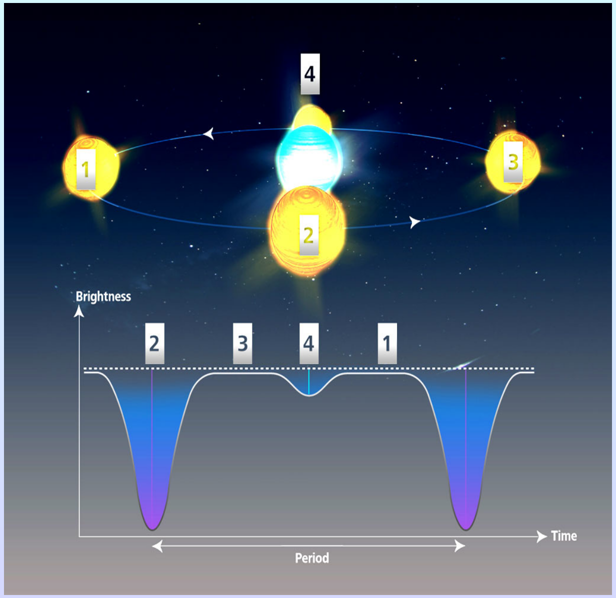
Obtaining Stellar Masses

There are basically 3 techniques that can be used to measure stellar masses

- Analysis of stellar absorption lines (pressure broadening, etc.) One measures (log) $g = G\,M / R^2 = 4\,\pi\,\sigma\,G\,M\,T^4 / L$
- Asteroseismology of non-radial pulsators
- Binary Stars



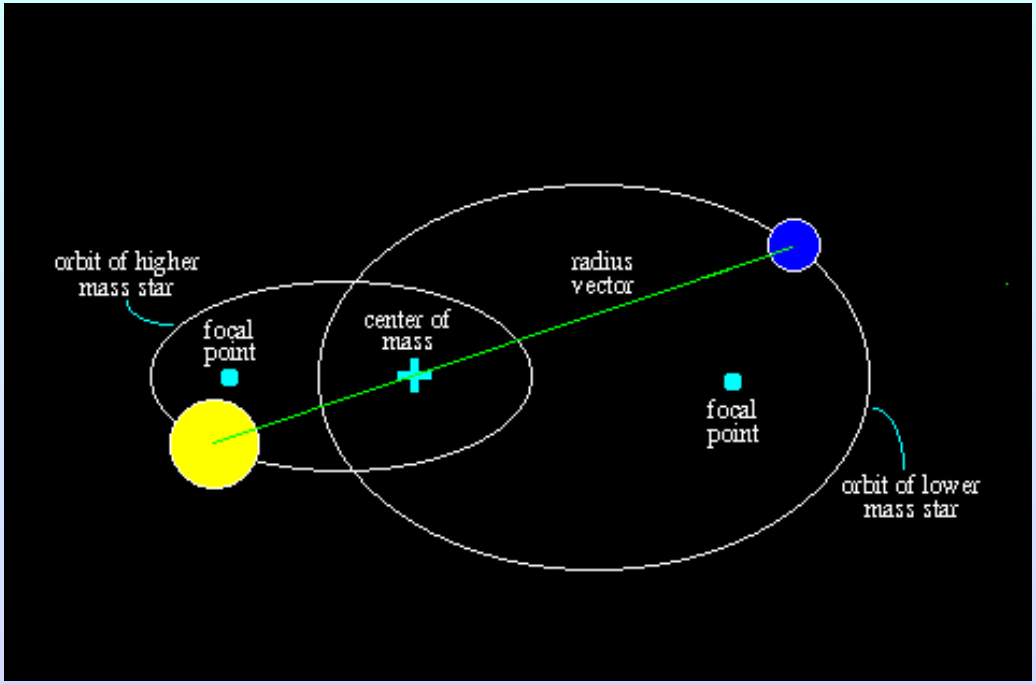




Visual Binaries

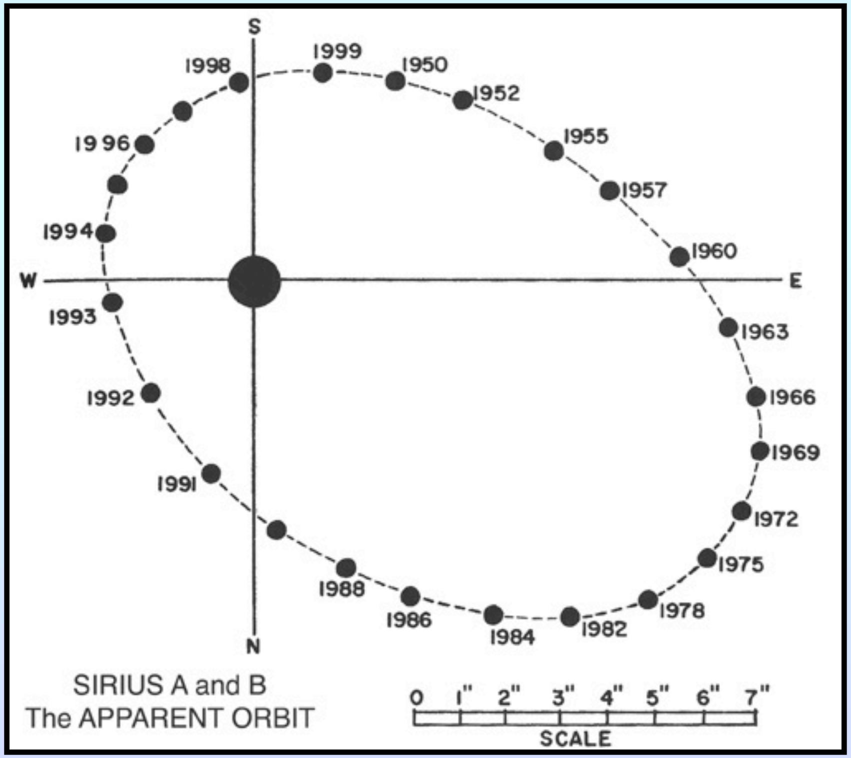
A perfect mass estimate of both stars is possible if:

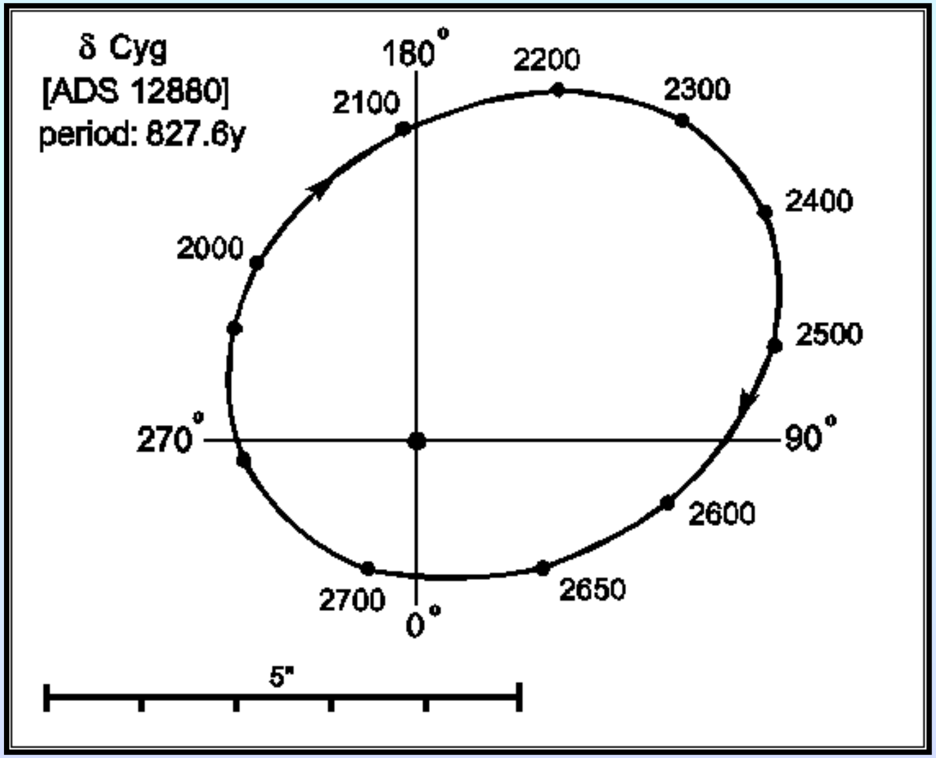
- Both stars are visible
- Their angular velocity is sufficiently high to allow a reasonable fraction of the orbit to be mapped
- The distance to the system is known (e.g., via parallax)
- The orbital plane is perpendicular to the line of sight



Periods of Visual Binaries

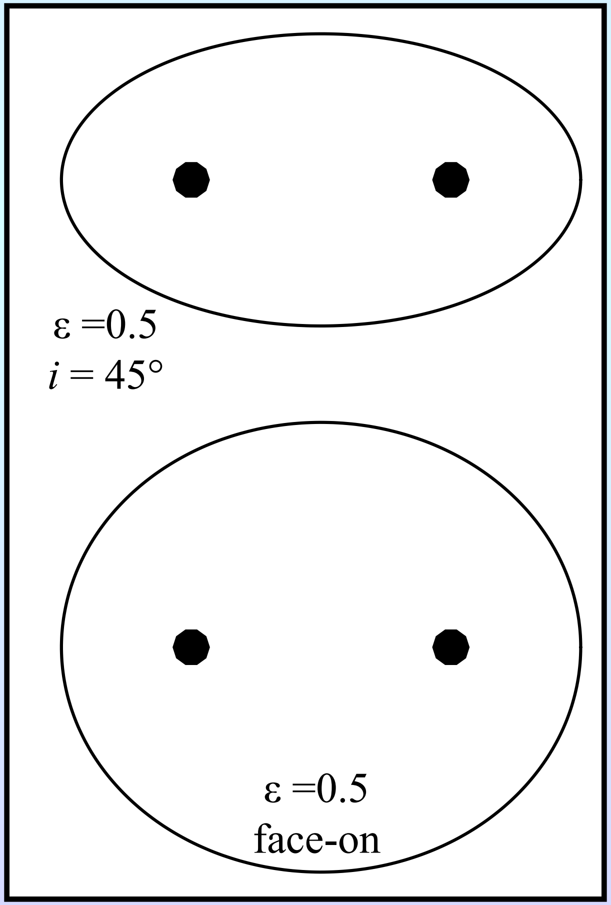
Note that for a solar-type visual binary at 10 pc, an $\alpha = 1''$ separation corresponds to a physical separation of 10 A.U. and a period of 33 years. More generally, for a distance d, $P \propto d^{3/2}\,\alpha^{3/2}$. So the periods can be *very* long.

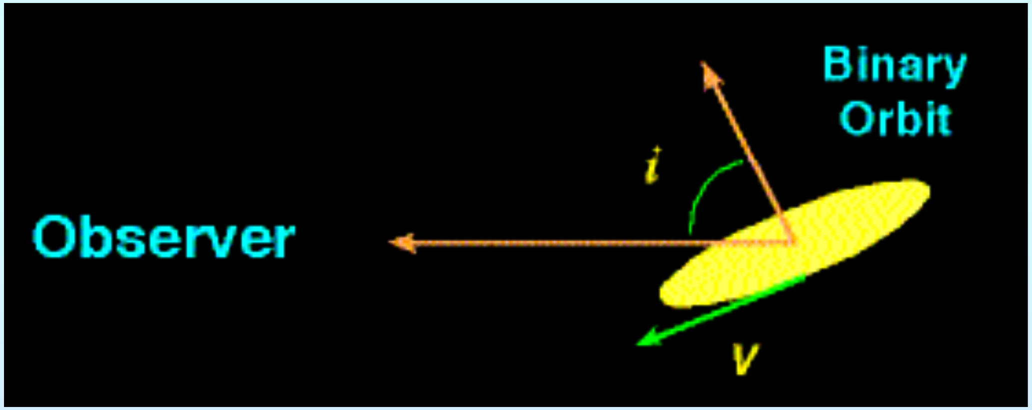




Visual Binaries

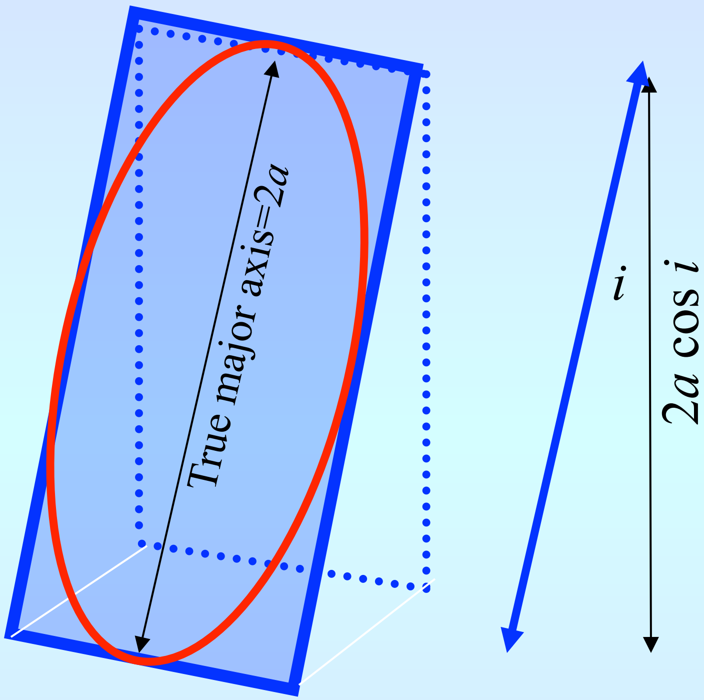
One difficulty with visual binaries is that, in general, you don't know the orbital inclination. (Is the orbit elliptical or an inclined circle?) There is the true ellipse and an apparent ellipse, but the focus of the true ellipse is not the focus of the apparent ellipse.

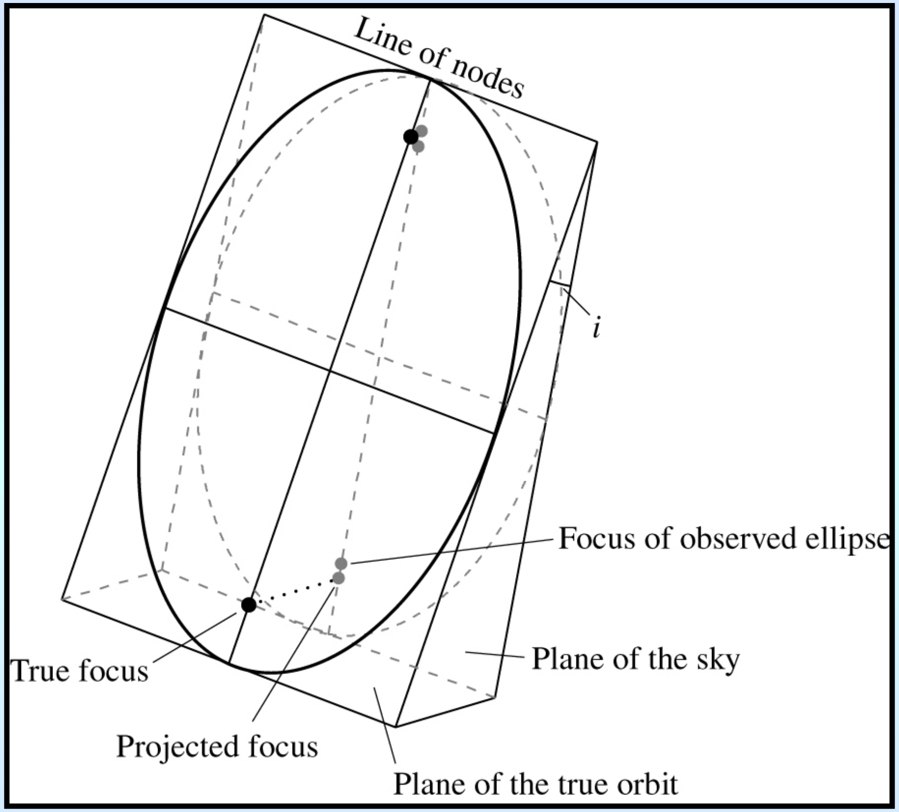




For visual binaries, you measure *a* cos *i*

Masses from Visual Binaries





The projection distorts the ellipse: the center of mass is not at the observed focus and the observed eccentricity is not the true eccentricity. Only with long-term, precise observations can you determine the true orbit.

Masses from Visual Binaries

The observed angular separation of a binary at a distance R will be $\alpha' = R\,\alpha_0 \cos i$. Relative masses will be unaffected by inclination

$$\boxed{\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1} = \frac{R\alpha_2}{R\alpha_1} = \frac{\alpha'_2 / R\cos i}{\alpha'_1 / R\cos i} = \frac{\alpha_2}{\alpha_1}}$$

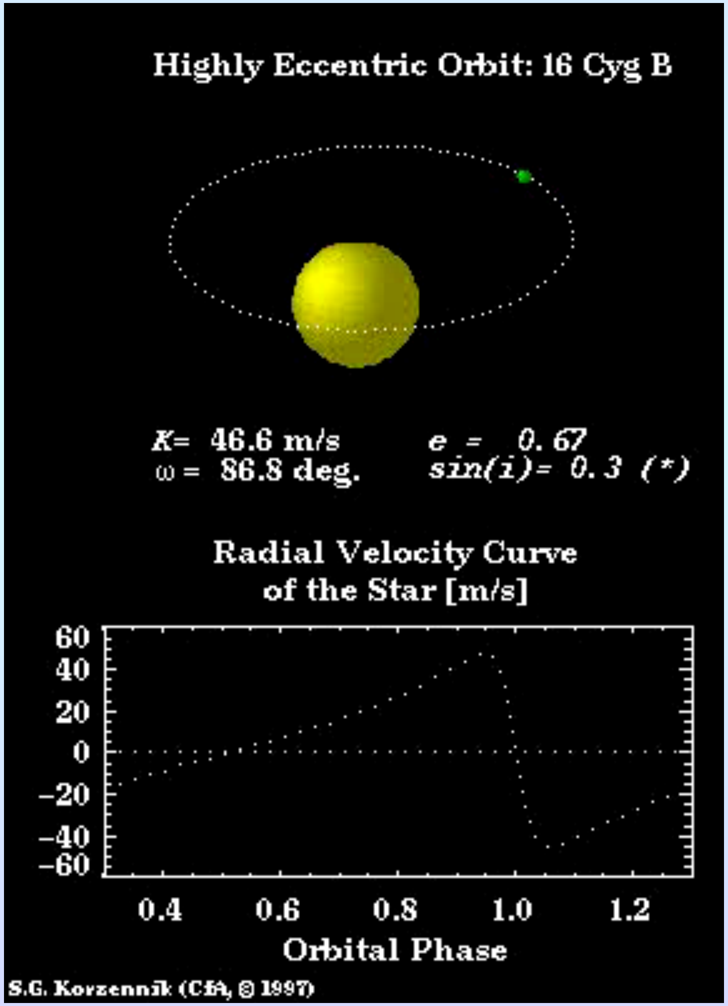
However, our measurement of total mass will only be a lower limit.

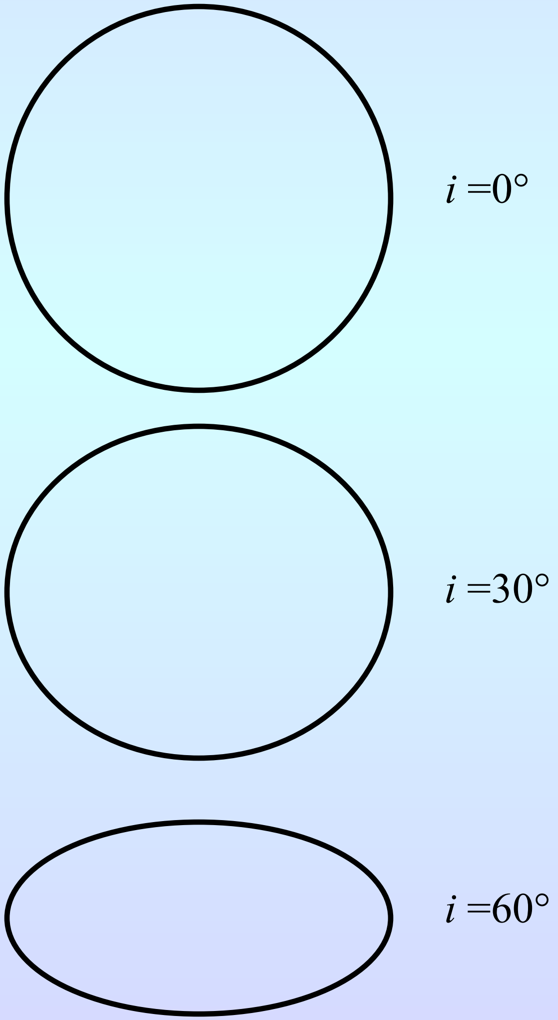
$$M_1 + M_2 = \frac{a^3}{kP^2} = \frac{(R\alpha')^3}{kP^2} = \frac{(R\alpha_0 / \cos i)^3}{kP^2} = \left(\frac{R}{\cos i}\right)^3 \frac{\alpha_0^3}{kP^2}$$

The masses derived from visual binaries will depend on distance, and $\cos^3 i$.

Spectroscopic Binaries

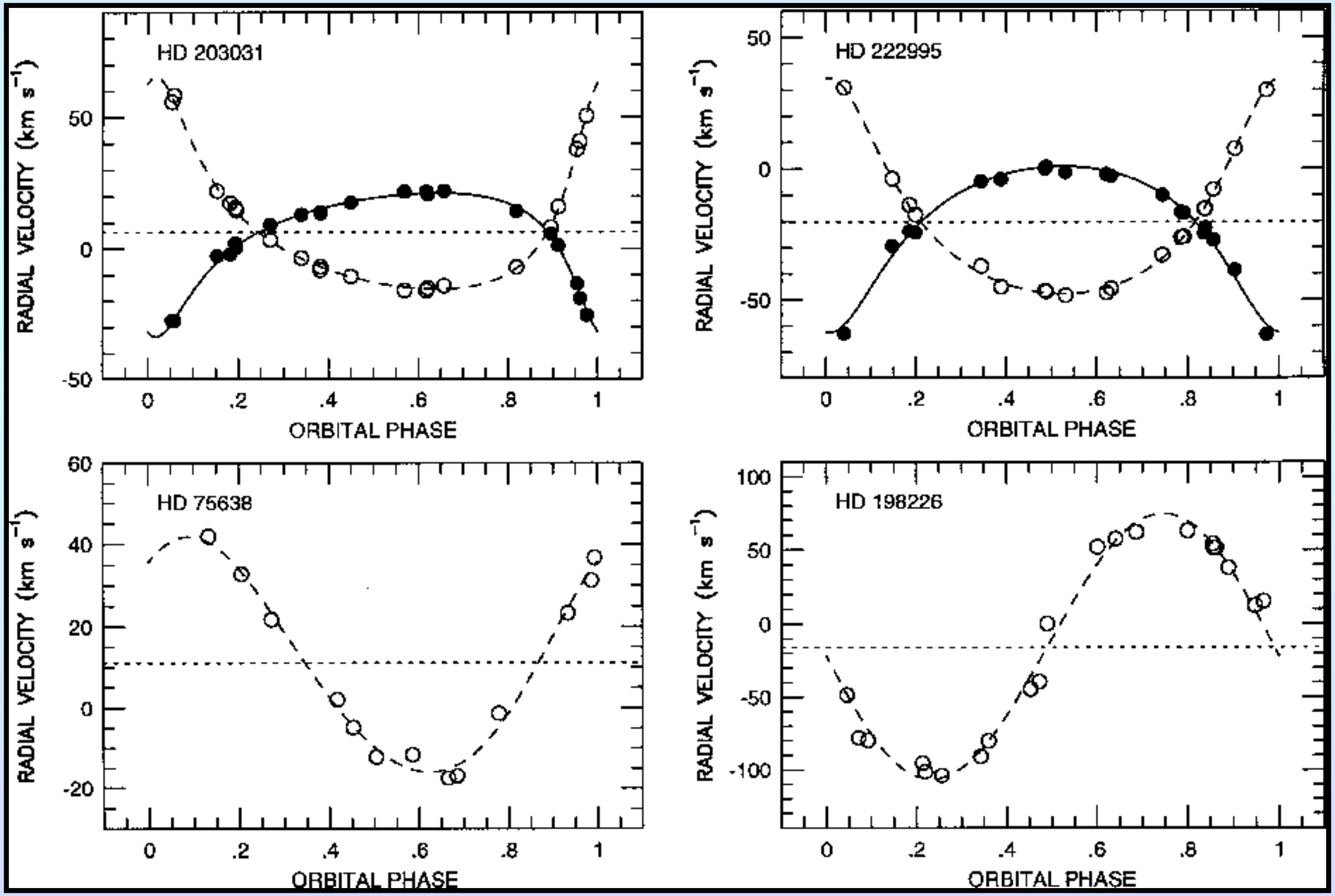
Orbits in the plane of the sky ($i = 0°$) show no radial velocity. In general, $v_{\mathrm{obs}} = v_{\mathrm{true}} \sin i$.





Separations of Spectroscopic Binaries

For a solar-type star at ~ 10 pc, a ~ 1 year period corresponds to a separation of 0.1", and a 1 km/s velocity corresponds to ~ 5 A.U. Most spectroscopic binaries are therefore unresolved.



The Mass Function

From Kepler's 3rd law, one can obtain a relation between the semi-amplitude of the velocity (K_1), the period, and the ratio of the masses, if the inclination of the system is known.

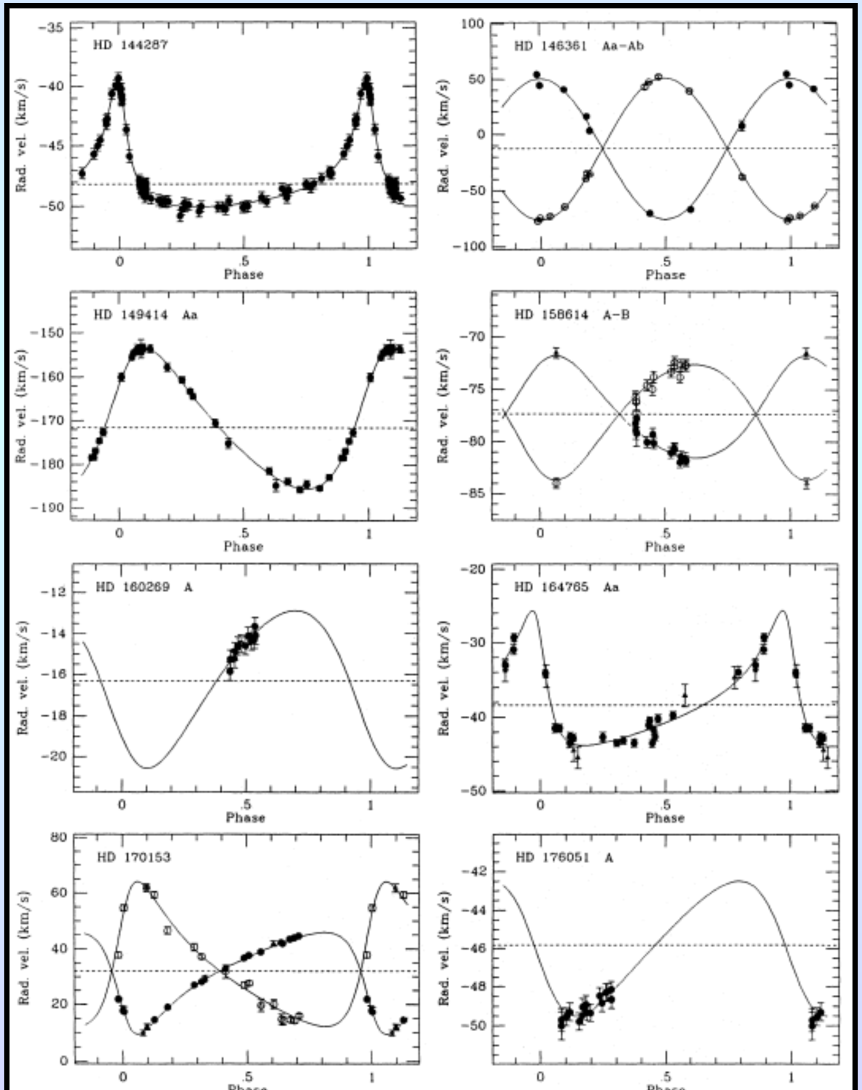
$$\frac{M_2^3}{(M_1+M_2)^2} = \frac{a_1^3}{k\,P^2} = \frac{P^3 K_1^3 \left(1-\varepsilon^2\right)^{3/2}}{(2\pi)^3 \sin^3 i} \cdot \frac{1}{k\,P^2}$$

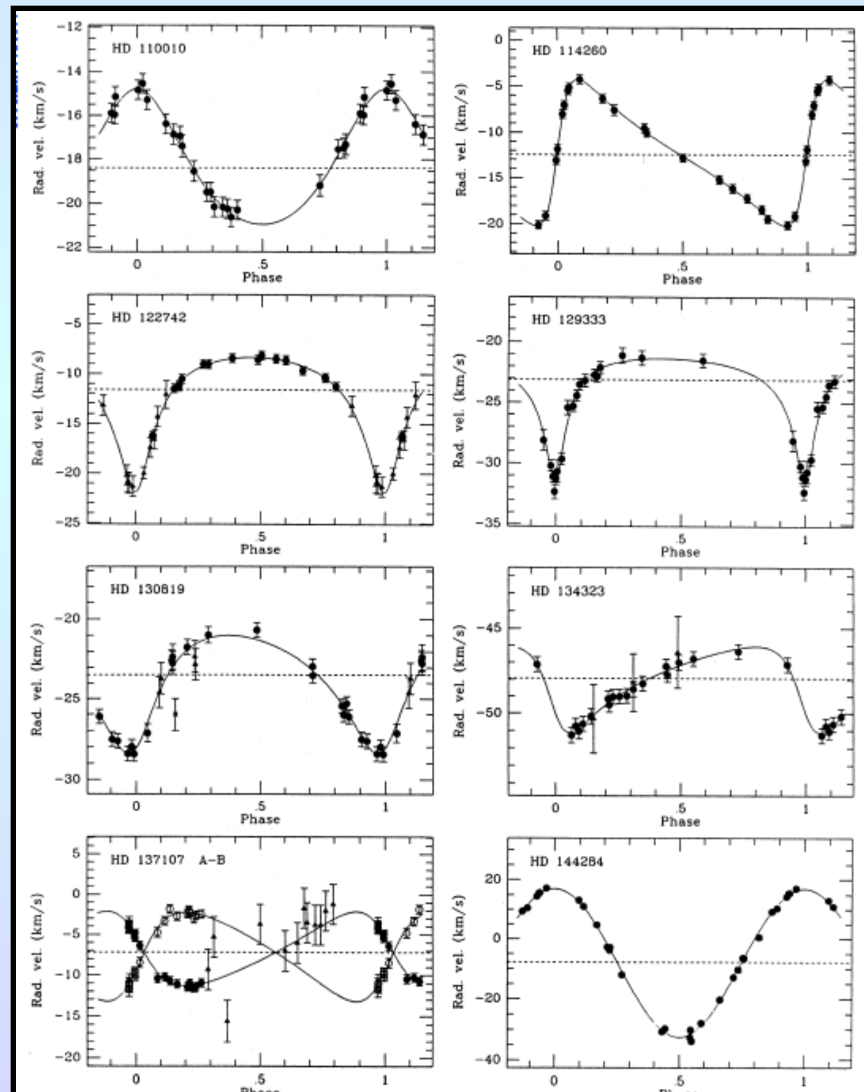
$$f(M_2) = \frac{M_2^3 \sin^3 i}{(M_2+M_1)^2} = \frac{P\,K_1^3 \left(1-\varepsilon^2\right)^{3/2}}{k\,(2\pi)^3}$$

For single-line spectroscopy binaries, it is impossible to determine individual masses, mass ratios, or even total mass. If both stars are seen, then the ratio of the masses can be found. But note: the results depend on $\sin^3 i$. Thus, to determine precise masses, one needs the system to be eclipsing.

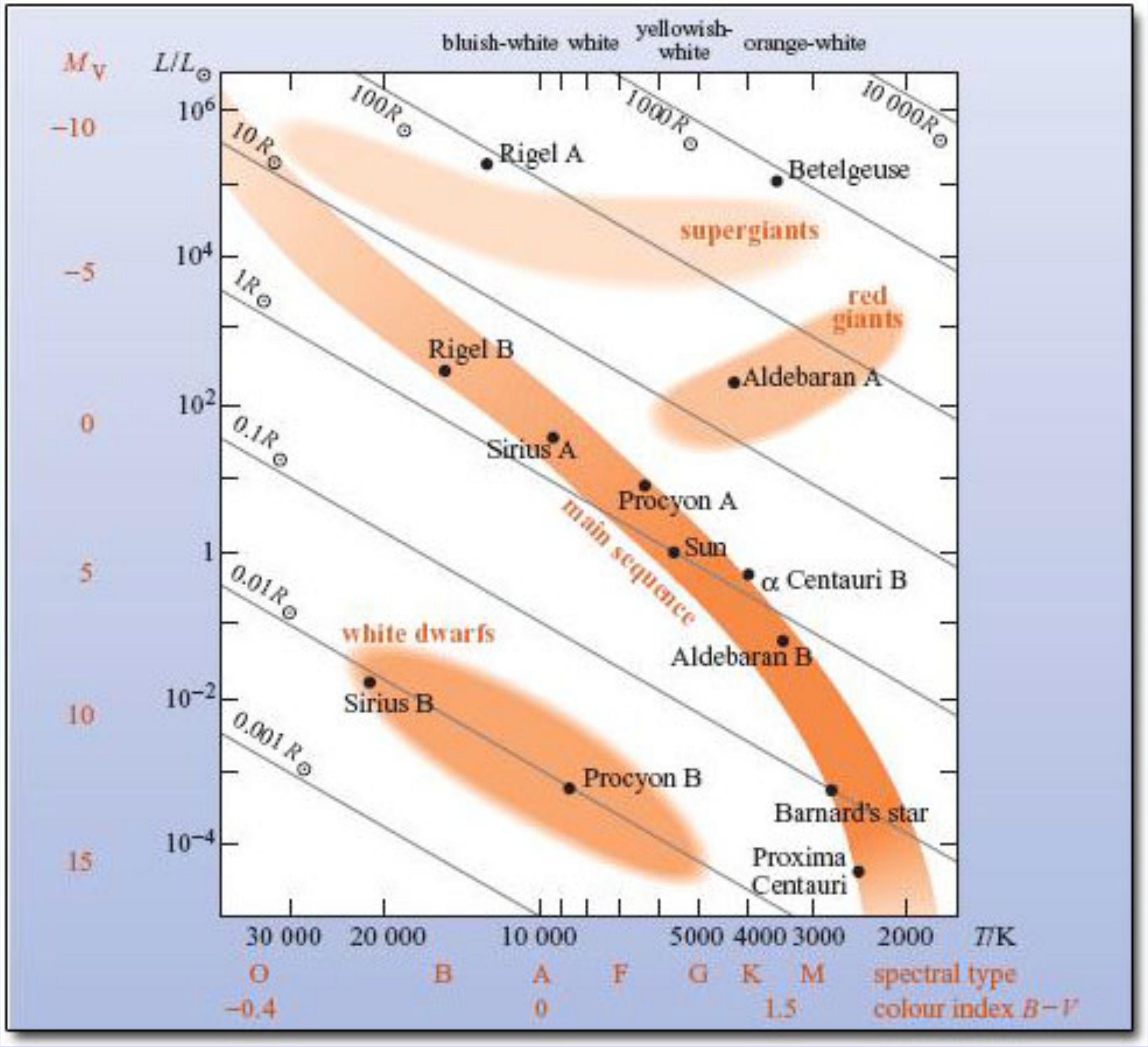
Radial Velocity Curves for Spectroscopic Binaries

Depending on the eccentricity, the radial velocity curve of a binary can have many shapes; the closer to sinusoidal, the closer ε = 0°.



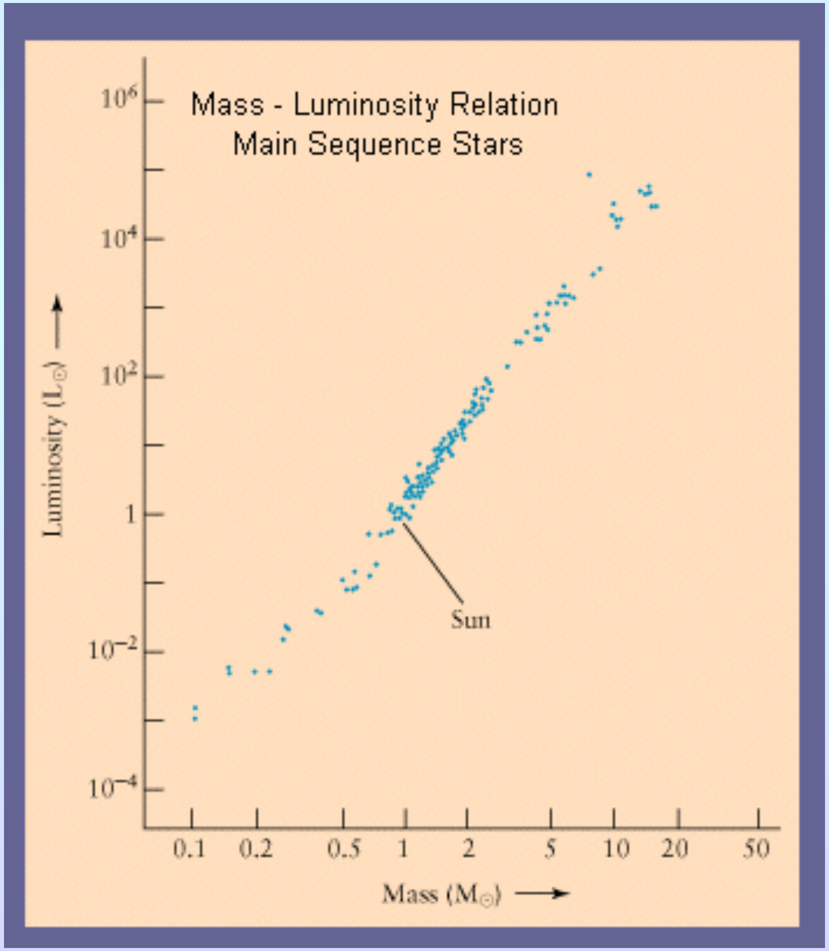


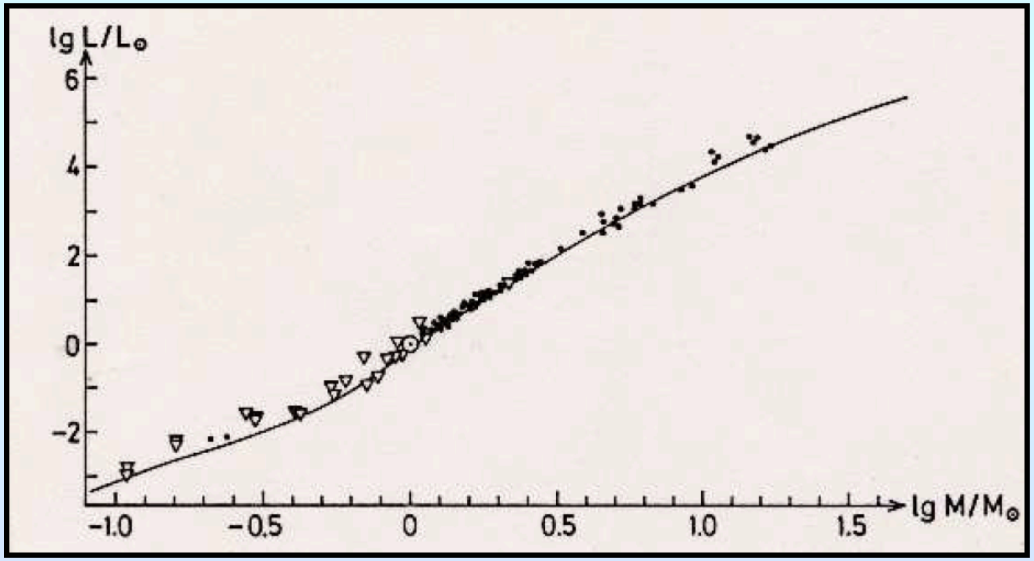
The HR Diagram with Iso-Radius Lines



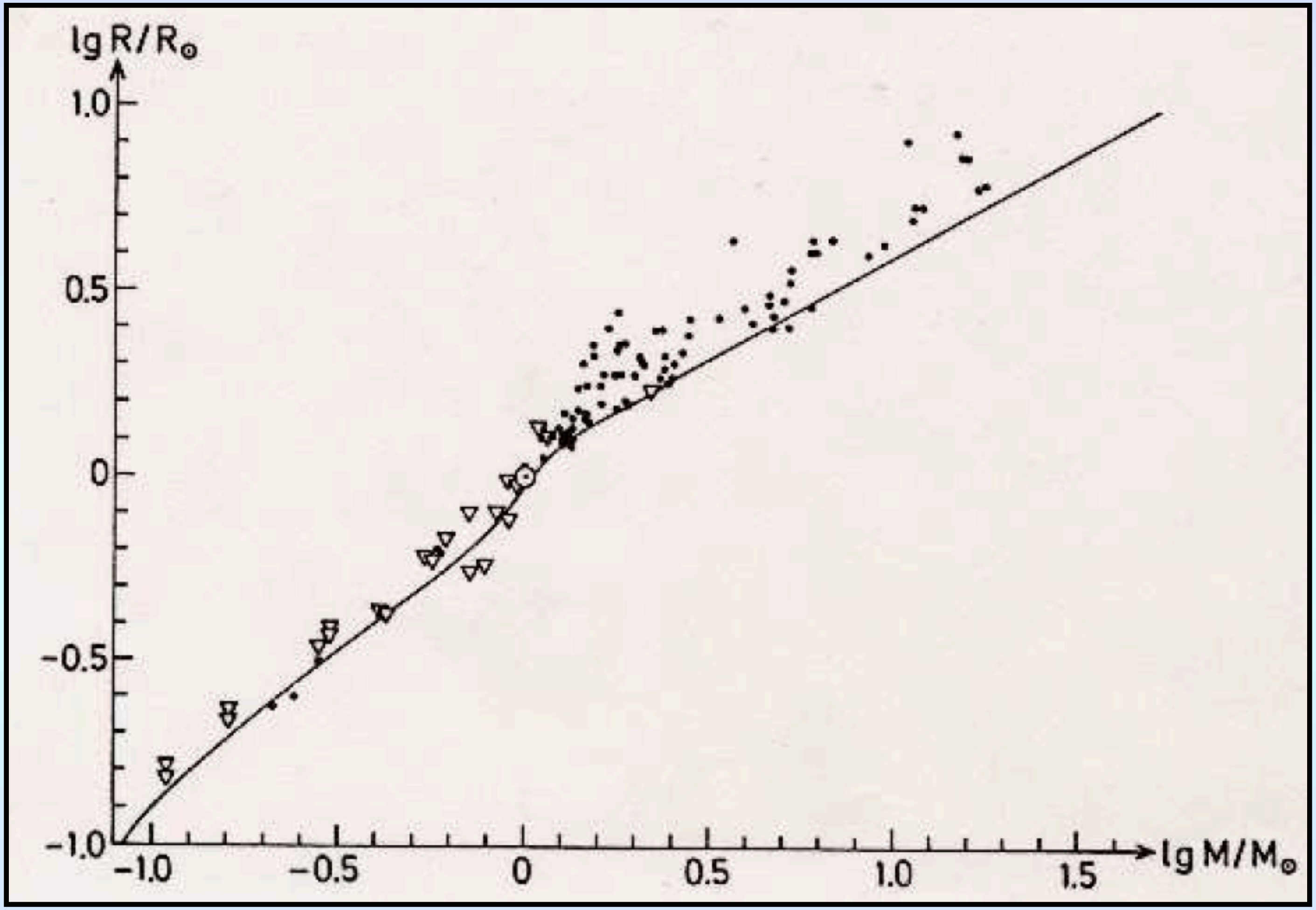
Stellar Masses and the Main Sequence

Measurements of main-sequence stars demonstrate that there is a mass-luminosity relationship, i.e., $L \propto M^{\eta}$. For $M > 1\ M_{\odot}$ $\eta \sim 3.88$, while at lower masses, the relation flattens out. A good rule-of-thumb is $L \propto M^{\eta}$, with $\eta \sim 3.5$.



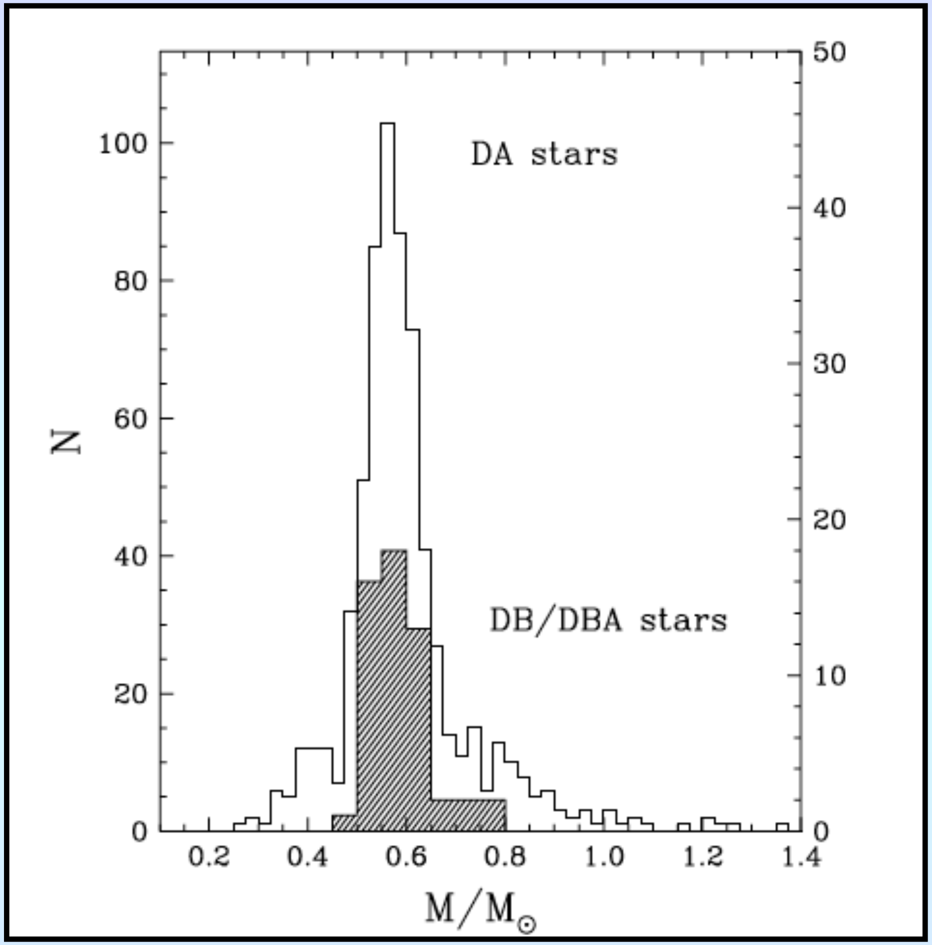


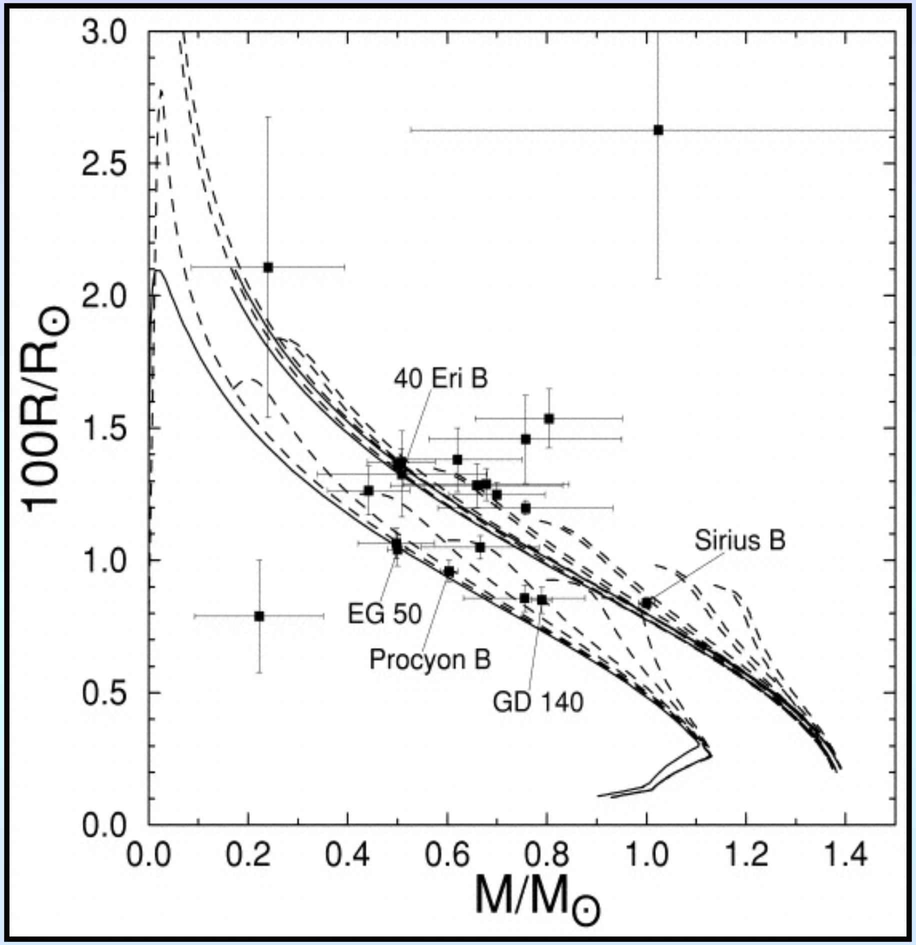
Main Sequence Mass-Radius Relation



There is also a mass-radius relation for main-sequence stars. When parameterized via a power law, $R \propto M^{\xi}$, $\xi \sim 0.57$ for $M > 1\ M_{\odot}$, and $\xi \sim 0.8$ for $M < 1\ M_{\odot}$.

Stellar Masses for White Dwarfs





The masses of white dwarf stars are all less than 1.4 $M_\odot$. Most are ~ 0.59 $M_\odot$.

There is also an inverse mass-radius relation for white dwarfs. The simple theory says $M \propto R^{\alpha}$, with $\alpha = -1/3$.

Star Clusters

Finally, we know from the main-sequence "turn-off" of star clusters that high-mass stars evolve more quickly than low-mass stars. This is easily seen from simple energy production arguments:

$$\tau \propto \frac{M}{L} \propto \frac{M}{M^{\alpha}} \propto M^{1-\alpha} \propto M^{-2.5}$$